When Exchange Rings are Von Neumann Regular

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Abstract: We study when exchange rings are von Neumann regular. An exchange ring R with primitive factors Artinian is von Neumann regular, if the Jacobson radical of any indecomposable homomorphic image of R is T-nilpotent, and if any indecomposable homomorphic image of R is semiprime. Every indecomposable semiprimitive factor ring of R is regular, if R is an exchange ring such that every left primitive factor ring of R is a ring of index at most n and if R has *nil*-property.

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§Introduction

In this paper, $J(R)$ denotes the Jacobson radical of a ring R. All modules are left R-modules. Regular always means von Neumann regular.

Let R be an associative ring with identity, and M a unitary left R-module. Crawley and Jonsson called M to have the exchange property if for every left R-module A and any two decompositions of A,

$$A = M' \oplus N = \oplus_{i \in I} A_i$$

where $M' \cong M$, there are submodules $A_i' \subseteq A_i$ such that

$$A = M' \oplus (\oplus_{i \in I} A_i').$$

It follows from the modular law that A_i' must be a direct summand of A_i for all i. Warfield [13] called a ring R an exchange ring if R has the exchange property as a left R-module. He proved that this definition is left and right symmetric. Many classes of rings are contained in this class of rings, for instance, we have the following relations: local rings are semiperfect rings, semiperfect rings are semiregular rings, and semiregular rings are exchange rings. Von Neumann regular rings are π-regular rings, π-regular rings are exchange rings. Artinian rings are perfect rings, perfect rings are strong π-regular rings, strong π-regular rings are strongly clean rings, strongly clean rings are clean rings, clean rings are exchange rings, see [3-6, 9-15].

Many authors have studied exchange rings with primitive factors Artinian. It was proved by Yu that an exchange ring with primitive factors Artinian is semi-strongly π-regular, if $R/J(R)$ is homomorphically semiprimitive (i.e., every ring homomorphic image of $R/J(R)$ has zero Jacobson radical) [15, Theorem 3.6]. In [6, Theorem 8], it was proved that the exchange ring R with primitive factors Artinian is a strongly π-regular ring, if the Jacobson radical of any homomorphic image of R is either T-nilpotent or locally nilpotent. It follows from Wu [14, Proposition 2.8 (1)]wt that an exchange ring R with primitive factors Artinian is regular if R is homomorphically semiprimitive. It was also proved by Wu in [14, Proposition 2.8 (2)]wt that for an exchange ring R with primitive factors Artinian, R is a von Neumann regular ring if every right primitive factor ring of R is flat as a left R-module. In Section 1, we prove that an exchange ring R with primitive factors Artinian is von Neumann regular, if the Jacobson radical of any indecomposable homomorphic image of R is right T-nilpotent, and if any indecomposable homomorphic image of R is semiprime. We also give some applications.

Recall that a ring R is called left weakly P-exchange ring if every left projective module over R has the finite exchange property. We call a ring R to be of bounded index or be of index at most n if there exists a number n such that $r^n = 0$ for every nilpotent element r of R. A ring R is called an I_0-ring if every left ideal of R that is not contained in $J(R)$ contains a nonzero idempotent. This property is left and right symmetric [12, Lemma 15.1]. It is well-known that R is a strongly π-regular ring and any prime factor ring of R is isomorphic to the ring of $n \times n$ matrices over a division ring, if R is a weakly P-exchange ring of bounded index, where $n \leq k$ if R has bounded index k [11, Theorem 4.11]. Recall that a ring without nonzero nilpotent ideals is called a semiprime ring, and a ring is called a prime ring if the product of any two of its nonzero ideals is not equal to zero. It is clear that the prime ring is semiprime. The intersection of all prime ideals of a ring R is called the prime radical of R. In Section 3, we prove that every indecomposable semiprimitive factor ring of R is von Neumann regular, if R is an exchange ring such that every left primitive factor ring of R is a ring of index at most n and R has nil-property.

§1.　Exchange Rings with Primitive Factors Artinian

It is well-known that a ring R is von Neumann regular if and only if the following conditions hold: R is a semiprime ring, every prime factor ring of R is a regular ring, and the union of every chain of semipime ideals of R is a semiprime ideal [8, Theorem 1.17]. For an exchange

ring we have the following result. Note that a subset I of a ring R is right T-nilpotent in case for every sequence a_1, a_2, $\ldots$ in I there is an n such that $a_n \ldots a_1 = 0$.

Theorem 1.1   Let R be an exchange ring with primitive factors Artinian. If the Jacobson radical of any indecomposable homomorphic image of R is right T-nilpotent and if any indecomposable homomorphic image of R is semiprime, then R is von Neumann regular.

Proof   Assume that R is not regular. There exists $a \in R$ such that a is not in aRa. Set

$$\mathcal{A} = \{I | \text{I is a two} - \text{sided ideal of} R \text{ and } a \notin (aRa + I)\}.$$

Since the zero ideal is in $\mathcal{A}$, $\mathcal{A}$ is not empty. Given any chain $\{I_\alpha, \ (\alpha \in \Lambda)\}$ in $\mathcal{A}$, where Λ is an indexed set, set $I = \cup_{\alpha \in \Lambda} I_\alpha$, then I is a two-sided ideal of R. If I is not in $\mathcal{A}$, $a \in aRa + I$. There must be an integer k such that $a \in aRa + I_k$, which is a contradiction. Thus the union of every chain in $\mathcal{A}$ is again in $\mathcal{A}$. By Zorn's Lemma, $\mathcal{A}$ contains a maximal element A.

We first claim that R/A is indecomposable as a ring. If not, then there exist ideals J, K of R such that $A \subsetneq J$, $A \subsetneq K$ and $R/A = J/A \oplus K/A$. And hence $J \cap K = A$ since $J/A \cap K/A = 0$. By definition of $\mathcal{A}$, there exist x, $y \in R$ such that $a - axa \in J$ and $a - aya \in K$. Then we have $a - a(x + y - xay)a = a - axa - (a - axa)ya = a - aya - ax(a - aya) \in J \cap K$. So $a - a(x + y - xay)a \in A$, which is a contradiction. Hence R/A is indecomposable as a ring.

Suppose that $J(R/A)$ is right T-nilpotent. If A is a prime ideal of R, we claim that $J(R/A) = 0$. Assume to the contrary that there exists a non-zero a in $J(R/A)$. Then there exist x_1, $x_2, \ldots \in R/A$ such that $ax_1 a \neq 0$, $ax_2 ax_1 a \neq 0, \ldots$, $ax_n \ldots ax_1 a \neq 0$, $\ldots$. Since $J(R/A)$ is T-nilpotent, we obtain a contradiction. Thus $J(R/A) = 0$. Since R is an exchange ring with primitive factor Artinian, so is R/A. By Nicholson [10, Proposition 1.4], every homomorphic image of an exchange ring is exchange. So R/A is an exchange ring. It is easy to see that exchange rings are I_0-rings. Using [15, Lemma 3.7], we see that R/A is simple Artinian, hence it is regular, which contradicts the choice of A. Then we assume that A is not prime. Consequently, we have two-sided ideals K, L of R such that $A \subsetneq K$, $A \subsetneq L$ and $KL \subseteq A$. Set $K' = \{r \in R | rL \subseteq A\}$, and set $L' = \{r \in R | K'r \subseteq A\}$. Since A is a semiprime ideal by hypothesis, the ideals K' and L' are semiprime. Since $(K' \cap L')^2 \subseteq K'L' \subseteq A$, $((K' \cap L')/A)^2 = 0$, i.e., $(K' \cap L')/A$ is a nilpotent ideal of R/A. Note that, since A is semiprime, $K' \cap L' \subseteq A$. It is clear that $K \subseteq K'$ and $L \subseteq L'$, hence the ideal K' and L' properly contain A. By the maximality of A, we have that $a \in aRa + K'$, $a \in aRa + L'$. Thus $a \in aRa + K' \cap L' \subseteq aRa + A$, a contradiction. Thus R is regular.   □

Example 1.2[9, *Example* 1.3]   The condition that the Jacobson radical of any indecomposable homomorphic image of R is T-nilpotent is not superfluous. Let Q be the field of all rationals and S the ring of all rationals with odd denominators. Let $R = \{(x_1, \ldots, x_n, s, s, \ldots) | n \geq 1, x_i \in Q, s \in S\}$. S has a unique maximal ideal (2), where p is odd. Thus $J(S) \neq 0$, and S is not regular. By the proof of Theorem 1.1, S is not T-nilpotent, since S is a prime ring. Since S is a homomorphic image of R, the condition that the Jacobson radical of any indecomposable homomorphic image of R is T-nilpotent is not satisfied, and R is not regular.

Corollary 1.3   Let R be a weakly P-exchange ring with primitive factors Artinian. If

any indecomposable homomorphic image of R is semiprime, then R is von Neumann regular.

Proof　　Let Q be an ideal of R. Since R is a weakly P-exchange ring, so is R/Q by [11, Proposition 4.1]. And by [10, Proposition 2.11] $J(R/Q)$ is T-nilpotent, then R is regular by Theorem 1.1.

Recall that a ring is called left (right) quasi-duo if every maximal left (right) ideal is two-sided. It is easy to prove that a left (right) quasi-duo ring is with left (right) primitive factors Artinian. We have the following corollary.

Corollary 1.4　　For a right or left quasi-duo exchange ring R, if $J(R/Q)$ is T-nilpotent for any ideal Q of R and if any homomorphic image of R is semiprime, then R is von Neumann regular.

Proof　　Since quasi-duo rings are with primitive factors Artinian, by Theorem 1.1, the assertion is true.

Corollary 1.5　　Let R be an exchange ring of bounded index. If the Jacobson radical of any indecomposable homomorphic image of R is T-nilpotent and if any indecomposable homomorphic image of R is semiprime, then R is von Neumann regular.

Proof　　By virtue of [16, Theorem 3], we know that R is an exchange ring with primitive factors Artinian, so we can check it applying Theorem 1.1.

Recall that a ring R is called an Abelian ring if all idempotents of R are central. We have the following corollary.

Corollary 1.6　　Let R be an Abelian exchange ring with $J(R) = 0$. If the Jacobson radical of any indecomposable homomorphic image of R is T-nilpotent and if any indecomposable homomorphic image of R is semiprime, then R is von Neumann regular.

Proof　　For an exchange ring R with zero Jacobson radical, idempoents are central iff R contains no nonzero nilpotent element [11, Lemma 4.10]. Thus R is abelian iff R is of bounded index 1. By Corollary 1.5, R is regular.

Corollary 1.7　　Let R be a ring satisfying one of the following conditions: (1) $R/J(R)$ is an exchange PI-ring; (2) for any finite or countably infinite many of orthogonal idempotents $e_1, e_2, \ldots$ and any $x_1, x_2, \ldots$ in R, there exists an integer $n > 0$, such that $e_1 x_1 e_2 x_2 \ldots e_n x_n \in J(R)$; (3) $R/J(R)$ is an exchange ring with finite orthogonal noncentral idempotent elements. If the Jacobson radical of any indecomposable homomorphic image of R is T-nilpotent and if any indecomposable homomorphic image of R is semiprime, then R is von Neumann regular.

Proof　　It follows from [14, Corollary 2.4], [14, Theorem 2.5], [14, Corollary 2.6] and Theorem 1.1.

Proposition 1.8　　Let R be a semiprimitive exchange ring with a finite set of orthogonal idempotents. Then R is von Neumann regular.

Proof By [5, Corollary 2], R is semiperfect. Since R is semiprimitive, R is semisimple. So R is regular since semisimple rings are regular.

Corollary 1.9   Let R be an exchange ring with a finite set of orthogonal idempotents. Then $R/J(R)$ is von Neumann regular.

Proof $R/J(R)$ is an exchange ring with a finite set of orthogonal idempotents, since idempotents can be lifted modulo $J(R)$. Since $R/J(R)$ is semiprimitive, by Proposition 1.8, it is regular.

It is well-known that von Neumann regular rings are π-regular, but there is a π-regular ring that is not regular [12, Proposition 23.4 (2)]. We have the following results, since π-regular rings are exchange rings.

Corollary 1.10   (1) Let R be a semiprimitive π-regular ring with a finite set of orthogonal idempotents. Then R is von Neumann regular.

(2) Let R be a semiprime π-regular ring with a finite set of orthogonal idempotents. Then R is von Neumann regular.

Corollary 1.11   (1) Let M be a module with the finite exchange property. If $J(End(M)) = 0$, and $End(M)$ has a finite set of orthogonal idempotents, then $End(M)$ is von Neumann regular.

(2) Let M be an indecomposable module with the finite exchange property. If $J(End(M)) = 0$, then $End(M)$ is von Neumann regular.

Proof   (1) It is clear.

(2) Since M is indecomposable, $End(M)$ has only two orthogonal idempotents. By (1), $End(M)$ is von Neumann regular.

§2.   Exchange Rings with Prime Factors Artinian

It is well-known that for a regular ring the following conditions are equivalent: each of its left (right) primitive factors is Artinian and each of its prime factors is Artinian [8, Theorem 6.2]. But for exchange ring it is not true. In fact, recall that Yu [15, Theorem 2.1] proved that the exchange ring with prime factors Artinian is strongly π-regular, but he also gave an example of an exchange ring with primitive factors Artinian which is not strongly π-regular [15, Example 3.5]yu1. Kaplansky once made the following conjecture: A ring R is von Neumann regular if and only if R is semiprime and each prime factor ring of R is von Neumann regular. However, the conjecture failed in general [8, Example 1.19]go. Luckily, the conjecture holds for commutative rings [7]. We have the following properties.

Proposition 2.1   Let R be a commutative exchange ring with prime factors Artinian. If R is semiprime and $J(R/P) = 0$ for any prime ideal P, then R is von Neumann regular.

Proof   By the Kaplansky's conjecture for the commutative ring R, it is regular iff it is semiprime and R/P is regular for any prime ideal P. It suffices to check that R/P is regular for any prime ideal P. Since R is an exchange ring with prime factors Artinian, So is R/P. Since every left primitive ring is a prime ring, R/P is a ring with primitive factors Artinian. It

is easy to check that exchange rings are I_0-rings. Since every prime ring is indecomposable as a ring, so is R/P. By [15, Corollary 3.7], R/P is simple Artinian, whence is regular.

Proposition 2.2   Let R be an exchange ring with prime factors Artinian. If all ideals of R are idempotent and $J(R/P) = 0$ for any prime P, then R is von Neumann regular.

Proof   Recall that a ring is regular iff all of its ideals are idempotent and all of its prime factors are regular [8, Corollary 1.18]. It is sufficient to check that any prime factor R/P is regular, which follows from the proof of Proposition 2.1.

§3.   Exchange Rings of Bounded Index

It is well-known that a regular ring R has index at most n, then all primitive factor rings of R are Artinian with length at most n [8, Corollary 7.10]. This is also true for exchange rings [16, Theorem 3]. We have the following results.

Theorem 3.1   Let R be an exchange ring of bounded index n. For any prime ideal P of R, if $J(R/P) = 0$, then R/P is von Neumann regular.

Proof   We first show that R/P contains at most n orthogonal idempotents for any prime ideal P of R.

Suppose that $\bar{e_1}, \bar{e_2}, \ldots, \bar{e_{n+1}}$ are nontrivial orthogonal idempotents in R/P. Since idempotents can be lifted modulo every ideal of an exchange ring [10, Corollary 1.3], we name the lifting orthogonal idempotents $e_1, e_2, \ldots, e_{n+1}$ of $\bar{e_1}, \bar{e_2}, \ldots, \bar{e_{n+1}}$ respectively, none of which lies in P. Since P is prime, there exist $x_1, x_2, \ldots, x_n$ in R such that $e_1 x_1 e_2 \notin P$, $e_1 x_1 e_2 x_2 e_3 \notin P, \ldots, e_1 x_1 e_2 x_2 e_3, \ldots e_n x_n e_{n+1} \notin P$. Set

$$r \equiv e_1 x_1 e_2 + e_2 x_2 e_3 + \ldots + e_n x_n e_{n+1}.$$

It is easy to check that $r^n = e_1 x_1 e_2 x_2 e_3 \ldots e_n x_n e_{n+1}$, and $r^{n+1} = 0$. Since R has bounded index n, $r^n = 0$. It is a contradiction, since $r^n \notin P$.

Exchange rings containing a finite set of orthogonal idempotents are semiperfect [5, Corollary 2]. since $J(R/P) = 0$, R/P is semisimple. Since semisimple rings are regular, so is R/P.

Corollary 3.2   Let R be an exchange ring of bounded index n. Then primitive factors are regular.

Proof   Since primitive rings are semiprimitive and prime, it is true following from Theorem 3.1.

Recall that an ideal P of R is called a minimal prime ideal of R if P is a minimal element of the set of all prime ideals of R. By Zorn's Lemma, Every prime ideal of R contains a minimal prime ideal of R.

We call a ring R has *nil*-property if the Jacobson radical of homomorphic image of R is a nil ideal. It is clear that the π-regular ring has *nil*-property.

Lemma 3.3   Let R be a semiprime ring of index at most n. If R has *nil*-property, then

R is semiprimitive.

Proof   Let N be the prime radical of R. Let $h : R \to R/N$ be the natural epimorphism. By [12, Remark 14.4], $h(R)$ is a semiprime ring of index at most n. By [12, Corollary 14.3], $h(R)$ does not have nonzero nilideals. Since $J(R)$ is a nilideal, $h(J(R))$ is a nilideal of $h(R)$. Then $h(J(R)) = 0$. Therefore $J(R) \subseteq N \subseteq J(R)$. By [12, Remark 3.7], $J(R) = 0$.

Lemma 3.4   Let R be an exchange ring of index at most n. If R has nil-property and if Q is a prime ideal of R, then R/Q is a simple Artinian ring of index at most n.

Proof Q contains a minimal prime ideal of R, named P. Set $\bar{R} \equiv R/P$. By [12, Remark 14.4], $\bar{R}$ is a prime ring of index at most n. By [12, Remark 14.5], $\bar{R}$ is an ring without infinite orthogonal idempotents. since every homomorphic image of an exchange ring is exchange, by [5, Corollary 2], $\bar{R}$ is semiperfect. By Lemma 3.3, $\bar{R}$ is semisimple. Since $\bar{R}$ is prime, $\bar{R}$ is simple. Thus R/Q is a quotient of R/P and is a simple Artinian ring.

Proposition 3.5   Let R be an exchange ring such that every left primitive factor ring of R is a ring of index at most n. If R has nil-property, then every indecomposable semiprimitive factor ring of R is von Neumann regular.

Proof   Let $\bar{R}$ be an indecomposable semiprimitive factor ring of R. Since every homomorphic image of an exchange ring is exchange and every exchange ring is I_0-ring, by [12, Proposition 15.8], $\bar{R}$ is a simple ring. since simple rings are primitive, and primitive rings are prime, $\bar{R}$ is a prime exchange ring of index at most n. Thus by Lemma 3.4, $\bar{R}$ is a simple Artinian ring of index at most n. Thus $\bar{R}$ is regular.

Acknowledgments

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