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Dyons of Unit Topological Charges in Gauged Skyrme Model

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Abstract: Dyons are an important family of topological solitons carrying both electric and magnetic charges and the presence of a nontrivial temporal component of the gauge field essential for the existence of electricity often makes the analysis of the underlying nonlinear equations rather challenging since the governing action functional assumes an indefinite form. In this work, developing a constrained variational technique, We establish an existence theorem for the dyon solitons in a Skyrme model coupled with $SO(3)$-gauge fields, formulated by Brihaye, Kleihaus, and Tchrakian. These solutions carry unit monopole and Skyrme charges.

Key words: Skyrme model; gauge fields; dyons; calculus of variations for indefinite action functionals; weak convergence

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§1. Introduction

Physical Background and Origins. Dyon is a hypothetical particle carrying both electric and magnetic charges. In contemporary physics, dyons and monopoles are relevant theoretical constructs for an interpretation of quark confinement [20, 23]. As early as in 1894, P. Curie [13] first formulated the concept of a magnetic monopole, a particle with only one magnetic pole, whose existence was widely suspected. In 1931, Dirac [15] explored the electromagnetic duality in the Maxwell equations and obtained a mathematical formalism of magnetic monopoles. It

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is sophisticated and highly challenging to obtain the existence of monopole and dyons in the Yang–Mills theory by mathematical methods.

As is well known, the Skyrme model is important for baryon physics [1, 9, 21] and gauge fields are often introduced to investigate inter-particle forces [2, 6-7, 11, 14, 16]. Motivated by these applications, here we are interested in the combined system consisting of the Skyrme model coupled with $SO(3)$ gauge fields originally presented by Brihaye, Kleihaus, and Tchrakian [6]. We may mention that dyons of the Georgi–Glashow model [22] and the $SO(3)$ gauged Skyrme model [7, 18] have been obtained recently. Combining these two models, we encounter a new system. The numerical study of the Skyrme dyon solution conducted in [6] provides us evidence of existence of the dyons. The purpose of this paper is to give an analytic proof for the existence of such solutions. The difficulty of the indefinite action functional still appears as stated. To overcome this, we need to obtain suitable uniform estimates for a minimizing sequence at singular boundary points and achieve strong convergence for the sequence of the negative terms as seen in Gao and Yang [18].

Mathematical Problem. Let a, f, g, h be real-valued functions of variable $x \in [0, \infty)$ and satisfying the two-point boundary condition

$$\left.\begin{array}{l} a(0) = 1,\ f(0) = g(0) = h(0) = 0; \\ a(\infty) = 0,\ f(\infty) = \theta_0,\ h(\infty) = 1,\ g(\infty) = q, \end{array}\right\} \tag{1.1}$$

where θ_0 and q are some parameters. Consider the energy density functions $\mathcal{E}_1$ and $\mathcal{E}_2$ given by

$$\begin{aligned} \mathcal{E}_1 = {} & \left(\frac{\mathrm{d}a}{\mathrm{d}x}\right)^2 + \frac{(a^2-1)^2}{2x^2} + \frac{1}{2}x^2\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 + a^2h^2 + \frac{\lambda}{4}x^2(h^2-1)^2 + \\ & \kappa a^2 \sin^2 f\left(\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right]^2 + \frac{a^2\sin^2 f}{x^2}\right) + \frac{\xi}{2}\left(x^2\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right]^2 + 2a^2\sin^2 f\right), \end{aligned} \tag{1.2}$$

$$\mathcal{E}_2 = \frac{1}{2}x^2\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 + a^2g^2, \tag{1.3}$$

where $\lambda, \kappa \geq 0$ and $\xi > 0$ are constants. We shall aim at obtaining a critical point of the *indefinite* action functional

$$L(a, f, g, h) = \int_0^\infty (\mathcal{E}_1 - \mathcal{E}_2)\,\mathrm{d}x, \tag{1.4}$$

subject to the boundary condition (1.1) and the finite-energy condition

$$E(a, f, g, h) = \int_0^\infty (\mathcal{E}_1 + \mathcal{E}_2)\,\mathrm{d}x < \infty. \tag{1.5}$$

Such a critical point is a spherically symmetric particle-like solution, of the equations of motion of the classical Skyrme model coupled with gauge fields taking values in the Lie algebra of the orthogonal group $SO(3)$, carrying unit topological charge and coined as 'dyon' in quantum field theory.

The rest of the paper is organized as follows. In Section 2, we review the $SO(3)$ gauged Skyrme model of Brihaye–Kleihaus–Tchrakian [6]. In the subsequent three sections, we establish our main existence theorem. Specifically, in Section 3, we prove the existence of a finite-energy critical point of the indefinite action functional by formulating and solving a constrained minimization problem as in [18]. In Section 4, we show that the critical point obtained in the previous section for the constrained minimization problem solves the original equations of motion by proving that the constraint does not give rise to an unwanted Lagrange multiplier problem. In Section 5, we study the properties of the solutions. In particular, we obtain some uniform decay estimates which allow us to describe the dependence of the ('t Hooft) electric charge on the asymptotic value of the electric potential function at infinity.

§2. Dyons in The Combined Skyrme Model

The field-theoretical model under consideration is an elegant combination of the classical Georgi–Glashow model and of the semi-locally gauged Skyrme model investigated previously in [3, 8]. In the model, the matter field is an R^3-valued Higgs field, Φ^α $(\alpha=1,2,3)$ over the (3+1)-dimensional Minkowski spacetime $\mathbb{R}^{3,1}$ of signature $(+---)$, of coordinates denoted by x^μ $(\mu=0,1,2,3)$. Thus, using A^α_μ to denote the real-valued gauge fields in a fixed representation of the gauge group $SO(3)$. We can express the induced gauge-covariant derivatives and gauge field strength tensors as

$$D_\mu\Phi^\alpha=\partial_\mu\Phi^\alpha+\varepsilon^{\alpha\beta\gamma}A^\beta_\mu\Phi^\gamma,\quad F^\alpha_{\mu\nu}=\partial_\mu A^\alpha_\nu-\partial_\nu A^\alpha_\mu+\varepsilon^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu, \tag{2.1}$$

where the late Greek letters $\mu,\nu,...=0,1,2,3$ label the Minkowski spacetime indices, while the early Greek letters $\alpha,\beta,\cdots=1,2,3$ label the indices of the algebra of the gauge group $SO(3)$. With the above notation, the Lagrangian density of the Georgi–Glashow model is given by

$$\mathcal{L}_{\mathrm{GG}}=-\frac{1}{4}\lambda_0^4|F^\alpha_{\mu\nu}|^2+\frac{1}{2}\lambda_1^4|D_\mu\Phi^\alpha|^2-\frac{1}{4}\lambda_2^4(\eta^2-|\Phi^\alpha|^2)^2 \tag{2.2}$$

The finite-energy condition implies that $\Phi=(\Phi^\alpha)$ maps the 2-sphere at the infinity of the space $\mathbb{R}^3$ into the 2-sphere $|\Phi|^2=\eta^2$ $(\eta>0)$ which naturally associated Φ with an integer class in $\pi_2(S^2)=\mathbb{Z}$. Such an integer is called the monopole number of the model and is denoted by Q_{M}, which is also referred to as the monopole charge.

In the classical Skyrme model, one is concerned with a map $\phi=(\phi^a)$ where $a=(\alpha,4)$ from the Minkowski spacetime into the standard 3-sphere so that $|\phi|^2=\sum_{a=1}^4(\phi^a)^2=1$ and ϕ^a $(a=1,2,3,4)$ may also be expressed as (ϕ^α,ϕ^4) $(\alpha=1,2,3)$. Our gauged Skyrme model is so gauged that there is the usual global $O(4)$ symmetry but there is a local $SO(3)$ symmetry imposed on the ϕ^α part. Therefore the gauge-covariant derivatives are semi-local and defined by

$$D_\mu\phi^\alpha=\partial_\mu\phi^\alpha+\varepsilon^{\alpha\beta\gamma}A^\beta_\mu\phi^\gamma,\quad D_\mu\phi^4=\partial_\mu\phi^4. \tag{2.3}$$

As a consequence, the semi-local gauged Skyrme model is governed by the Lagrangian density

$$\mathcal{L}_{O(4)}=-\frac{1}{4}\kappa_0^4|F^\alpha_{\mu\nu}|^2+\frac{1}{2}\kappa_1^2|D_\mu\phi^\alpha|^2-\frac{1}{8}\kappa_2^4|D_{[\mu}\phi^aD_{\nu]}\phi^b|^2 \tag{2.4}$$

Following [3, 8], we shall consider also the equations of motion coming from the coupled model resulting from (2.2) and (2.4) given by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_{\mathrm{GG}} + \mathcal{L}_{O(4)}. \tag{2.5}$$

Interestingly, the non-Abelian Higgs field Φ and the Skyrme scalar field ϕ are now elegantly coupled through a single Lie algebra $so(3)$-valued gauge field, $A_\mu = (A_\mu^\alpha)$.

Restricting to spherically symmetric static field configurations, we have

$$A_i^\alpha = \frac{a(r)-1}{r}\varepsilon_{i\alpha\beta}\hat{x}^\beta, \quad A_0^\alpha = g(r)\hat{x}^\alpha \tag{2.6}$$

$$\Phi^\alpha = \eta h(r)\hat{x}^\alpha \tag{2.7}$$

$$\phi^\alpha = \sin f(r)\hat{x}^\alpha, \quad \phi^4 = \cos f(r), \tag{2.8}$$

where $\hat{x} = x/r, r = |x|, x = (x^1, x^2, x^3)$. Notice that the real-valued functions $a(r), h(r), g(r)$ and $f(r)$ are dimensionless. It will be convenient to introduce a dimensionless radial variable

$$x = \eta r, \tag{2.9}$$

which should not be confused with that denoting a point in $\mathbb{R}^3$ before.

Substituting the Ansatze (2.6)–(2.8) into the static version of the equations of motion of the Lagrangian density (2.5), we arrive at following one-dimensional (radial) Lagrangian density, which is still denoted by the same letter $\mathcal{L}$, defined by

$$\mathcal{L} = \mathcal{E}_1 - \mathcal{E}_2, \tag{2.10}$$

where

$$\begin{aligned}\mathcal{E}_1 = {} & (a')^2 + \frac{(a^2-1)^2}{2x^2} + \frac{1}{2}x^2(h')^2 + a^2h^2 + \frac{\lambda}{4}x^2(h^2-1)^2 + \\ & \kappa a^2\sin^2 f\left((f')^2 + \frac{a^2\sin^2 f}{x^2}\right) + \frac{\xi}{2}\left(x^2(f')^2 + 2a^2\sin^2 f\right),\end{aligned} \tag{2.11}$$

$$\mathcal{E}_2 = \frac{1}{2}x^2(g')^2 + a^2g^2, \tag{2.12}$$

and $'$ denotes the differentiation $\frac{\mathrm{d}}{\mathrm{d}x}$, $\xi > 0, \lambda \geq 0$ and $\kappa \geq 0$, such that the associated Hamiltonian (energy) density is given by

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2. \tag{2.13}$$

It should be noted that, although there are six free coupling constants, $\lambda_0, \lambda_1, \lambda_2$, and $\kappa_0, \kappa_1, \kappa_3$ in the original Lagrangian action density (2.5), the radially symmetrically reduced action density (2.10) as seen in (2.11) and (2.12) contains only three free coupling constants, ξ, λ, κ, after some appropriate rescaling formulation. See [3, 8] for details.

The equations of motion of the original Lagrangian density (2.10) now become the variational equation

$$\delta L = 0, \tag{2.14}$$

of the static action

$$L(a,f,h,g)=\int_0^\infty \mathcal{L}\,\mathrm{d}x=\int_0^\infty (\mathcal{E}_1-\mathcal{E}_2)\,\mathrm{d}x, \tag{2.15}$$

which is indefinite. Explicitly, the equation (2.14) may be expressed in terms of the unknowns a,f,h,g as

$$\begin{aligned} a''=a\bigg(&\frac{a^2-1}{x^2}+h^2+\xi\sin^2 f+\kappa\sin^2 f(f')^2+\\ &\frac{\kappa a^2\sin^4 f}{x^2}-g^2\bigg), \end{aligned} \tag{2.16}$$

$$\begin{aligned} \Big(2\kappa a^2\sin^2 f f'+\xi x^2 f'\Big)'=&2\xi a^2\sin f\cos f+2\kappa a^2\sin f\cos f(f')^2+\\ &\frac{2\kappa a^4\sin^3 f\cos f}{x^2}, \end{aligned} \tag{2.17}$$

$$(x^2h')'=\quad h\big(2a^2+\lambda x^2(h^2-1)\big), \tag{2.18}$$

$$(x^2g')'=2a^2g. \tag{2.19}$$

We are to solve these equations under suitable boundary conditions. First we observe in view of the ansatz (2.6)–(2.8) that the regularity of the fields ϕ and A_μ imposes at $x=0$ the boundary condition

$$a(0)=1,\quad f(0)=0,\quad h(0)=0,\quad g(0)=0. \tag{2.20}$$

Furthermore, the finite-energy condition

$$E(a,f,h,g)=\int_0^\infty \mathcal{E}\,\mathrm{d}x=\int_0^\infty (\mathcal{E}_1+\mathcal{E}_2)\,\mathrm{d}x<\infty, \tag{2.21}$$

the non-triviality of the g-sector lead us to the boundary condition at $x=\infty$, given as

$$a(\infty)=0,\quad f(\infty)=\theta_0,\quad h(\infty)=1,\quad g(\infty)=q, \tag{2.22}$$

where $0<\theta_0<\frac{\pi}{2}$, $q>0$ (say) is a parameter, to be specified later, which defined the asymptotic value of the electric potential at infinity.

It can be directly checked that the above radially symmetric ansatz leads to the unit monopole charge, $Q_{\mathrm{M}}=1$. There is another topological quantity, however, called the Skyrme charge, that needs to be elaborated upon in some detail. Recall that ϕ maps $\mathbb{R}^3$ into S^3. In the classical situation without local symmetry, ϕ has a limiting position at infinity of $\mathbb{R}^3$ which enables us to compactify $\mathbb{R}^3$ into S^3 so that ϕ defines a map from S^3 into itself and is thus characterized by an integer called the Skyrme charge, Q_{S} in the homotopy group $\pi_3(S^3)$. In the semi-locally gauged situation, however, the gauge coupling complicates the asymptotic behavior at infinity in such a way that the aforementioned compactification breaks down due to a spontaneous symmetry breaking effect caused by gauge fields which makes the topological

Note that Q_{M} is a topological charge, in fact a homotopy invariant defined from the Higgs scalar, which should not be confused with the magnetic charge Q_m, to be addressed below, arising from the underlying electromagnetism.

characterization of the field configuration much more sophisticated. Nevertheless, formally, the same integral formula that defines the Skyrme charge has the same expression [18]:

$$Q_{\mathrm{S}} = -\frac{2}{\pi}\int_0^\infty \sin^2 f(x) f'(x)\,\mathrm{d}x. \tag{2.23}$$

In our case, Q_{S} is no longer an integer. In fact, applying the boundary condition (2.20) and (2.22) in (2.23), we obtain

$$Q_{\mathrm{S}} = Q_{\mathrm{S}}(\theta_0) = \frac{1}{\pi}\left(\frac{1}{2}\sin 2\theta_0 - \theta_0\right), \tag{2.24}$$

which is strictly decreasing for $\theta_0 \in [0, \pi/2]$.

Finally, the associated magnetic and electric charges of the solutions following the 't Hooft electromagnetism as calculated in [18] are given by $Q_m = 1$ and

$$Q_e = 2\int_0^\infty a^2(x) g(x)\,\mathrm{d}x, \tag{2.25}$$

respectively.

We can now state our main result regarding the existence of dyon solitons in the coupled George–Glashow and the gauged Skyrme model [6] as follows.

Theorem 2.1 For any parameters θ_0 and q satisfying

$$0 < \theta_0 < \frac{\pi}{2}, \quad 0 < q < 1, \tag{2.26}$$

the equations of motion of the minimally gauged Skyrme model defined by the Lagrangian density (2.10), have a static finite-energy spherically symmetric solution described by the ansatz (2.6)–(2.8) so that (a, f, h, g) satisfies the boundary conditions (2.20) and (2.22). Furthermore, such a solution enjoys the property that $f(x), g(x)$ are strictly increasing, and $a(x) > 0, 0 < f(x) < \theta_0$, $0 < g(x) < q$, $h(x) > 0$ for all $x > 0$. The solution carries a unit monopole charge $Q_{\mathrm{M}} = 1$, a continuous Skyrme charge Q_{S} given as a function of θ_0 by

$$Q_{\mathrm{S}}(\theta_0) = \frac{1}{\pi}\left(\frac{1}{2}\sin 2\theta_0 - \theta_0\right), \quad 0 < \theta_0 < \frac{\pi}{2}, \tag{2.27}$$

which may assume any value in the interval $(-\frac{1}{2}, 0)$, a unit magnetic charge $Q_m = 1$, and an electric charge given by the integral

$$Q_e = 2\int_0^\infty a^2(x) g(x)\,\mathrm{d}x, \tag{2.28}$$

which depends on q and approaches zero as $q \to 0$.

The above theorem will be established in the subsequent sections.

§3. Constrained Minimization Problem

In this section, we will prove the theorem 2.1 by using a indefinite variational process. In functional (2.15), the difficulty arises from the negative terms and it will be overcomed by solving a constrained minimization problem. And we will show the solution solve the equations (2.16)–(2.19) in next section.

To proceed, we introduce the admissible space for our variational problem as follows

$$\begin{aligned}\mathcal{A}=\{(a,f,h,g)|&a,f,h,g \text{ are continuous functions over } [0,\infty) \text{ which are absolutely}\\ &\text{continuous on any compact subinterval of } (0,\infty), \text{ satisfy the boundary condi-}\\ &\text{tions } a(0)=1, a(\infty)=0, f(0)=0, f(\infty)=\theta_0, h(\infty)=1, g(\infty)=q, \text{ and}\\ &\text{of finite-energy } E(a,f,h,g)<\infty\}\,.\end{aligned}$$

It is hard to find a critical point of the indefinite functional L directly in $\mathcal{A}$. Then we can deal with the negative term with respect to g independently, and then consider a,f,h for fixed g. Following this, we need a further restriction: we assume that (a,f,h,g) satisfies the constraint

$$\int_0^\infty (x^2g'G'+2a^2gG)\,\mathrm{d}x=0, \tag{3.1}$$

where G is an arbitrary test function satisfying $G(\infty)=0$ and

$$J(a,G)=\int_0^\infty (x^2(G')^2+2a^2G^2)\,\mathrm{d}x<\infty. \tag{3.2}$$

The constrained class $\mathcal{C}$ is defined to be

$$\mathcal{C}=\{(a,f,h,g)\in\mathcal{A}|\,(a,f,h,g) \text{ satisfies } (3.1)\}. \tag{3.3}$$

In the rest of this section, we shall focus on the following constrained minimization problem

$$\min\{L(a,f,h,g)|(a,f,h,g)\in\mathcal{C}\}\,. \tag{3.4}$$

Here we emphasize that the constraint (3.1) partially freezes the temporal component of the gauge field, which, in view of the 't Hooft electromagnetic formalism (?) of non-Abelian gauge field theory, partially freezes the electric sector of the system of equations of motion. By doing so, we are able to tackle the negative component arising in the indefinite Lagrangian action functional. We then show that the minimizer of the full (indefinite) Lagrangian action (but not the positive definite energy functional) subject to the constraint (3.1) will be a finite-energy critical point of the Lagrangian action, which is a classical solution of the original equations of motion.

Lemma 3.1 Assume $0<\theta_0<\pi/2$. For the problem (3.4), we may always restrict our attention to functions f satisfying $0\le f\le\pi/2$.

Proof It is easy to see that $L(a,f,h,g) = L(a,|f|,h,g)$. Hence we may assume $f \geq 0$ in the minimization problem. Noting $f(\infty) = \theta_0 < \frac{\pi}{2}$, if there is some $x_0 > 0$ such that $f(x_0) > \frac{\pi}{2}$, we see that there is an interval (x_1, x_2) with $0 \leq x_1 < x_0 < x_2 < \infty$ such that $f(x) > \frac{\pi}{2}$ $(x \in (x_1, x_2))$ and $f(x_1) = f(x_2) = \frac{\pi}{2}$. We now modify f by reflecting f over the interval $[x_1, x_2]$ with respect to the level $\frac{\pi}{2}$ to get a new function $\tilde{f}$ satisfying $\tilde{f}(x) = \pi - f(x)$ $(x \in [x_1, x_2])$ and $\tilde{f}(x) = f(x)$ $(x \notin [x_1, x_2])$. It follows that $L(a,f,h,g) = L(a,\tilde{f},h,g)$ again.

Lemma 3.2 The constrained admissible class $\mathcal{C}$ defined in (3.3) is non-empty. Furthermore, if $q > 0$ and $(a,f,h,g) \in \mathcal{C}$, we have $0 < g(x) < q$ for all $x > 0$ and that g is the unique solution to the minimization problem

$$\min\left\{ J(a,G) \,\middle|\, G(\infty) = q \right\}. \tag{3.5}$$

Proof To proceed, we rewrite the action functional as

$$L(a,f,h,g) = \int_0^\infty \mathcal{L}\,\mathrm{d}x = \int_0^\infty \mathcal{E}_1\,\mathrm{d}x - \int_0^\infty \mathcal{E}_2\,\mathrm{d}x \equiv F(a,f,h) - J(a,g).$$

Then any element $(a,f,h,g) \in \mathcal{C}$ may be obtained by first choosing suitable a, f, h such that $F(a,f,h) < \infty$ and then choosing a unique g such that $g(\infty) = q$ and g solves the problem (3.5).

In fact, the Schwartz inequality gives us the asymptotic estimate

$$|G(x) - q| \leq \int_x^\infty |G'(s)|\,\mathrm{d}s \leq x^{-\frac{1}{2}} \left(\int_x^\infty s^2 (G'(s))^2\,\mathrm{d}s \right)^{\frac{1}{2}} \leq x^{-\frac{1}{2}} J^{\frac{1}{2}}(a,G), \tag{3.6}$$

which indicates that the limiting behavior $G(\infty) = q$ can be preserved for any minimizing sequence of the problem (3.5). Hence (3.5) is solvable. In fact, it has a unique solution, say g, for any given function a, in view of the functional $J(a,\cdot)$ is strictly convex. Since $J(a,\cdot)$ is even, we have $g \geq 0$. By the maximum principle in (2.19), we conclude with $0 < g(x) < q$ for all $x > 0$. The uniqueness of the solution to (3.5), for given a, follows easily.

Lemma 3.3 For any $(a,f,h,g) \in \mathcal{C}$, $g(x)$ is nondecreasing for $x > 0$ and $g(0) = 0$.

Proof Lemma 3.2 shows that $0 < g(x) < q$. We claim that

$$\liminf_{x \to 0} x^2 |g'(x)| = 0. \tag{3.7}$$

Assume otherwise, then there are $\epsilon_0, \delta > 0$, such that $x^2|g'(x)| \geq \epsilon_0$ for $0 < x < \delta$, which contradicts the convergence of the integral $\int_0^\infty x^2 (g')^2 \mathrm{d}x$.

Then (3.7) implies that there is a sequence $\{x_k\}$, such that $x_k \to 0$ and $x_k^2 |g'(x_k)| \to 0$, as $k \to \infty$. Noting this fact and (2.19), we obtain

$$\begin{aligned} x^2 g'(x) &= x^2 g'(x) - \lim_{k\to\infty} x_k^2 g'(x_k) \\ &= \lim_{k\to\infty} \int_{x_k}^x (s^2 g'(s))'\,\mathrm{d}s = \int_0^x (s^2 g'(s))'\,\mathrm{d}s \end{aligned}$$

$$= \int_0^x 2a^2(s)g(s)\,\mathrm{d}s \ge 0, \quad x > 0. \tag{3.8}$$

Therefore $g'(x) \ge 0$ and $g(x)$ is nondecreasing. As a result, we see that there is number $g_0 \ge 0$ such that

$$\lim_{r\to 0} g(x) = g_0. \tag{3.9}$$

It is necessary to prove $g_0 = 0$. Otherwise, if $g_0 > 0$, we use the fact $a(0) = 1, x^2 g'(x) \to 0$ $(x \to 0)$ (see (3.7)), and L'Hopital's rule to obtain

$$2g_0 = 2\lim_{x\to 0} a^2(x)g(x) = \lim_{x\to 0}(x^2 g')' = \lim_{x\to 0}\frac{x^2 g'(x)}{x} = \lim_{x\to 0} xg'(x).$$

Then, there is a $\delta > 0$, such that

$$g'(x) \ge \frac{g_0}{x}, \quad 0 < x < \delta. \tag{3.10}$$

Integrating (3.10), we obtain

$$|g(x_2) - g(x_1)| \ge g_0 \left|\ln \frac{x_2}{x_1}\right|,$$

which contradicts the existence of limit stated in (3.9). Then $g_0 = 0$, and the lemma is proved.

Lemma 3.4 For any $0 < \theta_0 < \pi/2$, $0 < q < 1$, and $(a, f, h, g) \in \mathcal{C}$, we have the following partial coercive lower estimate

$$L(a,f,h,g) \ge \int_0^\infty \mathrm{d}x \left\{ (a')^2 + \frac{(a^2-1)^2}{x^2} + \frac{\lambda}{4}x^2(h^2-1)^2 + \frac{\xi}{2}[x^2(f')^2 + 2a^2\sin^2 f] + \kappa a^2 \sin^2 f\left[(f')^2 + 2\frac{a^2\sin^2 f}{2x^2}\right] + \frac{C_1}{2}x^2(h')^2 + C_2 a^2 h^2 \right\}, \tag{3.11}$$

where $C_1, C_2 > 0$ are constants depending on q only.

Proof For any $(a, f, h, g) \in \mathcal{C}$, set $g_1 = qh$. Then g_1 satisfies $g_1(\infty) = q$. As a result, we have

$$J(a, g_1) \ge J(a, g), \tag{3.12}$$

and thus,

$$\begin{aligned} L(a,f,h,g) &= F(a,f,h) - J(a,g) \ge F(a,f,h) - J(a,g_1) \\ &= \int_0^\infty \mathrm{d}x \left\{ (a')^2 + \frac{(a^2-1)^2}{x^2} + \frac{\lambda}{4}x^2(h^2-1)^2 + \frac{\xi}{2}[x^2(f')^2 + 2a^2\sin^2 f] + \right. \\ &\left. \kappa a^2\sin^2 f\left[(f')^2 + 2\frac{a^2\sin^2 f}{2x^2}\right] + \frac{1-q^2}{2}x^2(h')^2 + (1-q^2)a^2h^2 \right\}, \end{aligned} \tag{3.13}$$

which implies the lower estimate (3.11).

Lemma 3.5 Under the conditions stated in Theorem 2.1, the constrained minimization problem (3.4) has a solution.

Proof Using Lemma 3.4, we see that

$$\eta_0 = \inf\{L(a,f,h,g) \,|\, (a,f,h,g) \in \mathcal{C}\} \tag{3.14}$$

is well defined. Let $\{(a_n, f_n, h_n, g_n)\}$ be any minimizing sequence of (3.4). That is, $(a_n, f_n, h_n, g_n) \in \mathcal{C}$ and $L(a_n, f_n, h_n, g_n) \to \eta_0$ as $n \to \infty$. Without loss of generality, we may assume $L(a_n, f_n, h_n, g_n) \leq \eta_0 + 1$ (say) for all n. Applying (3.11) and the Schwartz inequality, we obtain

$$|a_n(x) - 1| \leq \int_0^x |a_n'(s)| \, \mathrm{d}s \leq x^{\frac{1}{2}} \left(\int_0^x (a_n'(s))^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \leq C x^{\frac{1}{2}} (\eta_0 + 1)^{\frac{1}{2}}, \tag{3.15}$$

$$|f_n(x) - \theta_0| \leq \int_x^\infty |f_n'(s)| \, \mathrm{d}s \leq x^{-\frac{1}{2}} \left(\int_x^\infty s^2 (f_n'(s))^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \leq C x^{-\frac{1}{2}} (\eta_0 + 1)^{\frac{1}{2}}, \tag{3.16}$$

$$|h_n(x) - 1| \leq \int_x^\infty |h_n'(s)| \mathrm{d}s \leq x^{-\frac{1}{2}} \left(\int_x^\infty s^2 (h_n'(s))^2 \mathrm{d}s \right)^{\frac{1}{2}} \leq C x^{-\frac{1}{2}} (\eta_0 + 1)^{\frac{1}{2}}. \tag{3.17}$$

where $C > 0$ is a constant independent of n. In particular, $a_n(x) \to 1$ uniformly as $x \to 0$, $f_n(x) \to \theta_0$ and $h_n(x) \to 1$ uniformly as $x \to \infty$.

For any (a_n, f_n, h_n, g_n), the function $G_n = q h_n$ satisfies $G_n(\infty) = q$. Then, by the definition of g_n and (3.11), we have

$$J(a_n, g_n) \leq J(a_n, G_n) = q^2 \int_0^\infty (r^2 (h_n')^2 + a_n^2 h_n^2) \, \mathrm{d}r \leq C L(a_n, f_n, h_n, g_n), \tag{3.18}$$

where $C > 0$ is a constant, which shows that $J(a_n, g_n)$ is bounded as well.

With the above preparation, we are now ready to study the limit of the sequence $\{(a_n, f_n, h_n, g_n)\}$.

We introduce the Hilbert space $(X, (\cdot, \cdot))$, where the functions in X are all continuously defined in $x \geq 0$ and vanish at $x = 0$ and the inner product $(\cdot, \cdot)$ is defined by

$$(\omega_1, \omega_2) = \int_0^\infty \omega_1'(x) \omega_2'(x) \, \mathrm{d}x, \quad \omega_1, \omega_2 \in X.$$

Since $\{a_n - 1\}$ is bounded in $(X, (\cdot, \cdot))$, we may assume without loss of generality that $\{a_n\}$ has a weak limit, say, a, in the same space,

$$\int_0^\infty a_n' \omega' \, \mathrm{d}x \to \int_0^\infty a' \omega' \, \mathrm{d}x, \quad \forall \omega \in X,$$

as $n \to \infty$.

Similarly, we consider the Hilbert space $(Y, (\cdot, \cdot))$ where the functions in Y are all continuously defined in $x > 0$ and vanish at infinity with the inner product $(\cdot, \cdot)$ defined by

$$(\omega_1, \omega_2) = \int_0^\infty x^2 \omega_1' \omega_2' \, \mathrm{d}x, \quad \omega_1, \omega_2 \in Y.$$

Noting that $\{f_n - \theta_0\}$, $h_n - 1$ and $\{g_n - q\}$ are bounded in $(Y, (\cdot, \cdot))$, we may assume that there are functions f, h, g with $f(\infty) = \theta_0, h(\infty) = 1, g(\infty) = q$, and $f - \theta_0, h - 1, g - q \in (Y, (\cdot, \cdot))$, such that

$$\int_0^\infty x^2 W_n' \omega' \, \mathrm{d}x \to \int_0^\infty x^2 W' \omega' \, \mathrm{d}x, \quad \forall \omega \in Y, \tag{3.19}$$

as $n \to \infty$, for $W_n = f_n - \theta_0, \omega = f - \theta_0$, $W_n = h_n - 1, \omega = h - 1$, and $W_n = g_n - q, \omega = g - q$, respectively.

In the sequel, we need to prove that the weak limit (a, f, h, g) of the minimizing sequence $\{(a_n, f_n, h_n, g_n)\}$ obtained above actually lies in $\mathcal{C}$. In other words, we need to show that (a, f, h, g)satisfies the boundary conditions and the constraint (3.1). Using the uniform estimates (3.6), (3.15)-(3.17), we conclude that $a(0) = 1, f(\infty) = \theta_0, h(\infty) = 1, g(\infty) = q$. Moreover, by Lemma 3.4, we see that $a \in W^{1,2}(0, \infty)$. Therefore $a(\infty) = 0$. To show $f(0) = 0$, we use (3.15) to get a $\delta > 0$ such that

$$|a_n(x)| \geq \frac{1}{2}, \quad x \in [0, \delta]. \tag{3.20}$$

Then, by (3.21), we obtain

$$\begin{aligned} \sin^2 f_n(x) &\leq 2 \int_0^x |\sin f_n(s) f_n'(s)| \, \mathrm{d}s \\ &\leq 4x^{\frac{1}{2}} \left(\int_0^x a_n^2(s) \sin^2 f_n(s) (f_n'(s))^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \\ &\leq 4\kappa^{-\frac{1}{2}} x^{\frac{1}{2}} L^{\frac{1}{2}}(a_n, f_n, h_n, g_n), \quad x \in [0, \delta]. \end{aligned} \tag{3.21}$$

Noting $0 \leq f_n \leq \frac{\pi}{2}$, we invert (3.22) to get the uniform estimate

$$0 \leq f_n(x) \leq C x^{\frac{1}{4}}, \quad x \in [0, \delta], \tag{3.22}$$

where $C > 0$ is independent of n. Letting $n \to \infty$ in (3.23), we obtain $f(0) = 0$ as desired. Then the total boundary conditions are verified.

Then, the next thing is to verify (3.1). To this end, it is sufficient to establish the following results,

$$\int_0^\infty (a_n^2 g_n - a^2 g) G \, \mathrm{d}x \to 0, \tag{3.23}$$

$$\int_0^\infty (x^2 g_n' - x^2 g') G' \, \mathrm{d}x \to 0, \tag{3.24}$$

for any test function G satisfying (3.2) and $G(\infty) = 0$, as $n \to \infty$.

Using the fact $G \in Y$ and (3.20), we conclude that (**) is valid.

To get (3.24), we rewrite

$$\int_0^\infty (a_n^2 g_n - a^2 g) G \, \mathrm{d}x = \int_0^{\delta_1} + \int_{\delta_1}^{\delta_2} + \int_{\delta_2}^\infty \equiv I_1 + I_2 + I_3, \tag{3.25}$$

for some positive constants $0 < \delta_1 < \delta_2 < \infty$, and we begin with

$$I_1 = \int_0^{\delta_1} (a_n^2 - a^2) g_n G \, \mathrm{d}x + \int_0^{\delta_1} a^2 (g_n - g) G \, \mathrm{d}x \equiv I_{11} + I_{12}. \tag{3.26}$$

Noting (3.15) and (3.18), we see that there is a small $\delta > 0$ such that $g_n \in L^2(0, \delta)$ and there holds the uniform bound

$$\|g_n\|_{L^2(0,\delta)} \leq K, \tag{3.27}$$

for some constant $K > 0$. Therefore, we may assume $g_n \to g$ weakly in $L^2(0,\delta)$ as $n \to \infty$. In particular, $g \in L^2(0,\delta)$ and $\|g\|_{L^2(0,\delta)} \le K$. Besides, since in (3.2), the function a satisfies $a(0) = 1$, we have $G \in L^2(0,\delta)$ when $\delta > 0$ is chosen suitable small. Then, using (3.15) and taking $\delta_1 \le \delta$, we have

$$|I_{11}| \le \int_0^{\delta_1} |a_n^2 - a^2||g_n G|\,\mathrm{d}x \le \int_0^{\delta_1} \left(|a_n^2 - 1| + |a^2 - 1|\right)|g_n G|\,\mathrm{d}x \le CK\delta^{\frac{1}{2}}\|G\|_{L^2(0,\delta)}, \quad (3.28)$$

where $C > 0$ is a constant independent of n. Therefore, for any $\varepsilon > 0$, we can take $\delta_1 > 0$ sufficiently small to assure $|I_{11}| < \varepsilon$. On the other hand, since $g_n \to g$ weakly in $L^2(0,\delta)$ and $G \in L^2(0,\delta)$, we have $I_{12} \to 0$ as $n \to \infty$.

Since $\{a_n\}$ and $\{g_n\}$ are bounded sequences in $W^{1,2}(\delta_1,\delta_2)$, using the compact embedding $W^{1,2}(\delta_1,\delta_2) \mapsto C[\delta_1,\delta_2]$, we see that $a_n \to a$ and $g_n \to g$ uniformly over $[\delta_1,\delta_2]$ as $n \to \infty$. Thus $I_2 \to 0$ as $n \to \infty$.

To estimate I_3, we recall that $\{J(a_n, g_n)\}$ is bounded by (3.18), $g_n(x) \to q$ uniformly as $n \to \infty$ by (3.6), and $G(x) = \mathrm{O}(x^{-\frac{1}{2}})$ as $x \to \infty$ by (3.2). In particular, since $q > 0$, we may choose $x_0 > 0$ sufficiently large so that

$$|g(x)| \ge \frac{q}{2}, \quad \inf_n |g_n(x)| \ge \frac{q}{2}, \quad x \ge x_0. \quad (3.29)$$

From the above, we arrive at

$$|I_3| \le \int_x^\infty \left(|a_n^2 g_n| + |a^2 g|\right)|G|\,\mathrm{d}s \le Cx^{-\frac{1}{2}} \int_x^\infty \frac{2}{q}(a_n^2 g_n^2 + a^2 g^2)\,\mathrm{d}s, \quad (3.30)$$

where $x \ge x_0$ (cf. (3.30)) and $C > 0$ is a constant. From (3.18), we see that for any $\varepsilon > 0$ we may choose δ_2 large enough to get $|I_3| < \varepsilon$.

Summarizing the above discussion, we see that

$$\limsup_{n\to\infty} \left|\int_0^\infty (a_n^2 g_n - a^2 g) G\,\mathrm{d}x\right| \le 2\varepsilon, \quad (3.31)$$

which proves the desired conclusion(*). Thus, the claim $(a, f, h, g) \in \mathcal{C}$ follows.

To show that (a, f, h, g) is a solution of (3.4), we need to establish

$$\eta_0 = \liminf_{n\to\infty} L(a_n, f_n, h_n, g_n) \ge L(a, f, h, g). \quad (3.32)$$

It is difficult to get this fact due to the negative terms in the functional L. To overcome this, we may rewrite the Lagrange density (2.10) as

$$\mathcal{L}(a, f, h, g) = \mathcal{L}_0(a, f, h) - \mathcal{E}_0(a, g), \quad (3.33)$$

where

$$\begin{aligned}\mathcal{L}_0(a, f, h) = {} & (a')^2 + \frac{(a^2-1)^2}{2x^2} + \frac{1}{2}x^2(h')^2 + \frac{\lambda}{4}x^2(h^2-1)^2 + \\ & \kappa a^2 \sin^2 f\left[(f')^2 + \frac{a^2 \sin^2 f}{2x^2}\right] + \frac{\xi}{2}[x^2(f')^2 + 2a^2\sin^2 f] +\end{aligned}$$

$$a^2(h-1)^2+2a^2(h-1)+a^2(1-q^2) \tag{3.34}$$

$$\mathcal{E}_0(a,g)=\frac{1}{2}x^2(g')^2+a^2(g-q)^2+2a^2(g-q)q, \tag{3.35}$$

Therefore, to establish (3.33), we need to show that

$$\liminf_{n\to\infty}\int_0^\infty \mathcal{L}_0(a_n,f_n,h_n)\,\mathrm{d}x\geq\int_0^\infty \mathcal{L}_0(a,f,h)\,\mathrm{d}x, \tag{3.36}$$

$$\lim_{n\to\infty}\int_0^\infty \mathcal{E}_0(a_n,g_n)\,\mathrm{d}x=\int_0^\infty \mathcal{E}_0(a,g)\,\mathrm{d}x. \tag{3.37}$$

We first prove (3.38). Since both (a_n,g_n) and (a,g) satisfy (3.1), i.e.,

$$\int_0^\infty(x^2g_n'G'+2a_n^2g_nG)\,\mathrm{d}x=0,\quad \int_0^\infty(x^2g'G'+2a^2gG)\,\mathrm{d}x=0, \tag{3.38}$$

letting $G=g-g_n$ in the above equations and subtracting them, we obtain

$$\begin{aligned}\int_0^\infty x^2(g_n'-g')^2\mathrm{d}x&=2\int_0^\infty(a_n^2g_n-a^2g)(g-g_n)\mathrm{d}x\\&=\int_0^{\delta_1}+\int_{\delta_1}^{\delta_2}+\int_{\delta_2}^\infty\equiv I_1+I_2+I_3,\end{aligned} \tag{3.39}$$

where $0<\delta_1<\delta_2<\infty$.

To estimate I_1, we need to get some uniform estimate for the sequence $\{g_n\}$ as $x\to 0$. Using (3.15), we see that for any $0<\gamma<\frac{1}{2}$ (say) there is a $\delta>0$ such that

$$2a_n^2(x)\geq(2-\gamma),\quad x\in[0,\delta]. \tag{3.40}$$

We consider the comparison function of the form

$$\sigma(x)=Cx^{1-\gamma},\quad x\in[0,\delta],\quad C>0. \tag{3.41}$$

Then

$$(x^2\sigma')'=(1-\gamma)(2-\gamma)\sigma<2a_n^2(x)\sigma,\quad x\in[0,\delta]. \tag{3.42}$$

Since g_n solve (3.5) in Lemma 3.2, we have

$$(x^2(g_n-\sigma)')'>2a_n^2(x)(g_n-\sigma),\quad x\in[0,\delta]. \tag{3.43}$$

Choose $C>0$ in (3.42) large enough so that $C\delta^{1-\gamma}\geq q$. Noting $g_n<q$ (Lemma 3.2), we have $(g_n-\sigma)(\delta)<0$ and $(g_n-\sigma)(0)=0$. In view of these boundary conditions and applying the maximum principle to (3.44), we see that $g_n(x)<\sigma(x)$ for all $x\in(0,\delta)$. Or, more precisely, we have

$$0<g_n(x)<\left(\frac{q}{\delta^{1-\gamma}}\right)x^{1-\gamma},\quad 0<x<\delta. \tag{3.44}$$

Which implies that the weak limit g of $\{g_n\}$ satisfies the same estimate. Thus, from the uniform estimates (3.15) and (3.45), we see that for any $\varepsilon>0$ there is some $\delta_1>0$ $(\delta_1<\delta)$ such that $|I_1|<\varepsilon$.

Moreover, from (3.6) and (3.30), we get

$$\begin{aligned}|I_3| &\le 2\int_{\delta_2}^{\infty}(a_n^2 g_n + a^2 g)(|g_n - q| + |g - q|)\,\mathrm{d}x\\ &\le \frac{4}{q}(|g_n(\delta_2) - q| + |g(\delta_2) - q|)\int_0^{\infty}(a_n^2 g_n^2 + a^2 g^2)\,\mathrm{d}x\\ &\le \frac{2}{q}\delta_2^{-\frac{1}{2}}\left(J^{\frac{1}{2}}(a_n, g_n) + J^{\frac{1}{2}}(a, g)\right)(J(a_n, g_n) + J(a, g)),\end{aligned} \tag{3.45}$$

which may be made small than ε when $\delta_2 > 0$ is large enough due to (3.18).

Furthermore, since $a_n \to a$ and $g_n \to g$ in $C[\delta_1, \delta_2]$, we see that $I_2 \to 0$ as $n \to \infty$.

From the above results regarding I_1, I_2, I_3 in (3.40), we obtain the strong convergence

$$\lim_{n\to\infty}\int_0^{\infty} x^2(g_n' - g')^2\,\mathrm{d}x = 0. \tag{3.46}$$

In particular, we obtain

$$\lim_{n\to\infty}\int_0^{\infty} x^2(g_n')^2\,\mathrm{d}x = \int_0^{\infty} x^2(g')^2\,\mathrm{d}x. \tag{3.47}$$

We can also prove that

$$\lim_{n\to\infty}\int_0^{\infty}\left(a_n^2(g_n - q)^2 + 2a_n^2(g_n - q)q\right)\,\mathrm{d}x = \int_0^{\infty}\left(a^2(g - q)^2 + 2a^2(g - q)q\right)\,\mathrm{d}x. \tag{3.48}$$

In fact, we have the fact that $\{(a_n, f_n, h_n, g_n)\}$ is bounded in $W^{1,2}_{\mathrm{loc}}(0, \infty)$. Therefore the sequence is convergent in $C[\alpha, \beta]$ for any pair of numbers, $0 < \alpha < \beta < \infty$. Noting that $a_n(x) \to 1$ and $g_n(x) \to 0$ as $x \to 0$ uniformly, with respect to $n = 1, 2, \cdots$, we see that $a_n \to a$ and $g_n \to g$ uniformly over any interval $[0, \beta]$ $(0 < \beta < \infty)$. Hence, using this result and the uniform estimate (3.6), we get (3.49).

Then (3.38) follows from (3.48) and (3.49).

On the other hand, using the uniform estimate (3.17), we also have

$$\lim_{n\to\infty}\int_0^{\infty} a_n^2(h_n - 1)^2 + 2a_n^2(h_n - 1)\,\mathrm{d}x = \int_0^{\infty} a^2(h - 1)^2 + 2a^2(h - 1)\,\mathrm{d}x. \tag{3.49}$$

Finally, using (3.50), and the condition $q < 1$, i.e., $1 - q^2 > 0$, we get (3.37) and the proof of the lemma is complete.

§4. Fulfillment of The Governing Equations

Let (a, f, h, g) be the solution of (3.4) obtained in the previous section. Due to the negative terms in L, it is not obvious say (a, f, h, g) satisfies the governing equations (2.16)–(2.19). Since we have solved a constrained minimization problem, we need to prove that the Lagrange multiplier problem does not arise as a result of the constraint, which would otherwise alter the original equations of motion. In fact, since the constraint (3.1) involves a and g only and (3.1) immediately gives rise to (2.19), we see that all we have to do is to verify the validity of (2.16) because (2.17) and (2.18) are the f-equation and h-equation respectively (3.1) does not involve f and h explicitly.

To proceed, we take $\widetilde{a}\in C_0^1$. For any $t\in\mathbb{R}$, there is a unique corresponding function g_t such that $(a+t\widetilde{a},f,h,g_t)\in\mathcal{C}$ and that g_t smoothly depends on t. Set

$$g_t=g+\widetilde{g}_t,\quad \widetilde{G}=\left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{g}_t\right)\bigg|_{t=0}. \tag{4.1}$$

Since $(a+t\widetilde{a},f,h,g_t)|_{t=0}=(a,f,h,g)$ is a minimizing solution of (3.4), we have

$$\begin{aligned}0=&\frac{\mathrm{d}}{\mathrm{d}t}L(a+t\widetilde{a},f,h,g_t)\bigg|_{t=0}\\ =&\int_0^\infty \mathrm{d}x\bigg\{2a'\widetilde{a}'+\frac{(a^2-1)a\widetilde{a}}{x^2}+2ah^2\widetilde{a}+2\xi\sin^2 fa\widetilde{a}+2\kappa\sin^2 f(f')^2a\widetilde{a}\\ &\frac{2\kappa}{x^2}\sin^4 fa^3\widetilde{a}-2ag^2\widetilde{a}\bigg\}-\int_0^\infty\mathrm{d}x\bigg\{r^2g'\widetilde{G}'+2a^2g\widetilde{G}\bigg\}\\ \equiv& I_1-I_2.\end{aligned}\tag{4.2}$$

It is clear that the vanishing of I_1 implies (2.16) so that it suffices to prove that I_2 vanishes. To this end, from (3.1), we only need to show that $\widetilde{G}$ satisfies the same conditions required of G in (3.1).

In (3.1), using the replacements $a\mapsto a+t\widetilde{a}, g\mapsto g_t, G\mapsto\widetilde{g}_t$, we have

$$\int_0^\infty\left(x^2g_t'\widetilde{g}_t'+2(a+t\widetilde{a})^2g_t\widetilde{g}_t\right)\mathrm{d}x=0. \tag{4.3}$$

Or, with $g_t=g+\widetilde{g}_t$, we have

$$\int_0^\infty\left(x^2(g'+\widetilde{g}_t')\widetilde{g}_t'+2a^2(g+\widetilde{g}_t)\widetilde{g}_t+2t^2\widetilde{a}^2g_t\widetilde{g}_t+4ta\widetilde{a}g_t\widetilde{g}_t\right)\mathrm{d}x=0. \tag{4.4}$$

Recall that $\int_0^\infty\left(x^2g'\widetilde{g}_t'+2a^2g\widetilde{g}_t\right)\mathrm{d}x=0$. Using (4.4) and the Schwartz inequality, we have

$$\begin{aligned}&\int_0^\infty(x^2(\widetilde{g}_t')^2+2a^2\widetilde{g}_t^2)\,\mathrm{d}x=\left|2t\int_0^\infty(t\widetilde{a}^2+2a\widetilde{a})g_t\widetilde{g}_t\,\mathrm{d}x\right|\\ \le&|2t|\left(|2t|\int_0^\infty\widetilde{a}^2g_t^2\mathrm{d}x+\frac{1}{|2t|}\int_0^\infty a^2\widetilde{g}_t^2\,\mathrm{d}x\right)+2t^2\int_0^\infty\widetilde{a}^2|g_t|\,|\widetilde{g}_t|\,\mathrm{d}x\\ =&4t^2\int_0^\infty\widetilde{a}^2g_t^2\,\mathrm{d}x+\int_0^\infty a^2\widetilde{g}_t^2\,\mathrm{d}x+2t^2\int_0^\infty\widetilde{a}^2|g_t|\,|\widetilde{g}_t|\,\mathrm{d}x.\end{aligned}\tag{4.5}$$

Applying the bounds $0\le g,g_t\le q$ and the relation $\widetilde{g}_t=g_t-g$ in (4.5), we have

$$\begin{aligned}\int_0^\infty(x^2(\widetilde{g}_t')^2+a^2\widetilde{g}_t^2)\,\mathrm{d}x&\le 4t^2\int_0^\infty\widetilde{a}^2g_t^2\,\mathrm{d}x+2t^2\int_0^\infty\widetilde{a}^2|g_t|\,|\widetilde{g}_t|\,\mathrm{d}x\\ &\le 8q^2t^2\int_0^\infty\widetilde{a}^2\,\mathrm{d}x.\end{aligned}\tag{4.6}$$

As a consequence, we get

$$\int_0^\infty \left(x^2 \left(\frac{\widetilde{g}_t'}{t} \right)^2 + a^2 \left(\frac{\widetilde{g}_t}{t} \right)^2 \right) \mathrm{d}x \leq 8q^2 \int_0^\infty \widetilde{a}^2 \, \mathrm{d}x, \quad t \neq 0. \tag{4.7}$$

Applying $\widetilde{g}_t(\infty) = 0$, the Schwartz inequality and (4.7), we have for $t \neq 0$ the estimate

$$\left| \frac{\widetilde{g}_t}{t}(x) \right| \leq \int_x^\infty \left| \frac{\widetilde{g}_t'(s)}{t} \right| \mathrm{d}s \leq x^{-\frac{1}{2}} \left(\int_x^\infty s^2 \left(\frac{\widetilde{g}_t'}{t} \right)^2 \mathrm{d}s \right)^{\frac{1}{2}} \leq 2\sqrt{2} q \|\widetilde{a}\|_{L^2(0,\infty)}.$$

Letting $t \to 0$ in (4.7) and (4.8), we obtain $J(a, \widetilde{G}) < \infty$ and $\widetilde{G}(x) = \mathrm{O}(x^{-\frac{1}{2}})$ (for x large). In particular, $\widetilde{G}(\infty) = 0$ and $\widetilde{G}$ indeed satisfies all conditions required in (3.1) for G. Hence I_2 vanishes in (4.2). Therefore, the equation (2.16) has been fulfilled.

§5. Properties of The Solution Obtained

In this section, we investigate the properties of the solution (a, f, h, g).

Lemma 5.1 The solution (a, f, h, g) enjoys the properties $a(x) > 0$, $h(x) > 0$, $0 < g(x) < q$, $0 < f(x) < \theta_0$, and both $f(x)$ and $g(x)$ are strictly increasing, for any $x > 0$.

Proof From Lemmas 3.1 and 3.2, we see that $0 \leq g \leq q$ and $0 \leq f \leq \frac{\pi}{2}$. Besides, it is obvious that $a \geq 0$ since both (2.11) and (2.12) are even in a.

If $a(x_0) = 0$ for some $x_0 > 0$, then x_0 is a minimizing point and $a'(x_0) = 0$. Using the uniqueness of the solution to the initial value problem consisting of (2.16) and $a(x_0) = a'(x_0) = 0$, we get $a \equiv 0$ which contradicts $a(0) = 1$. Thus, $a(x) > 0$ for all $x > 0$. Similarly, we can prove that $f(x) > 0, g(x) > 0$ for all $x > 0$. Since (3.8) is valid, we see that $g(x)$ is strictly increasing. In particular, $g(x) < q$ for all $x > 0$.

Lemma 3.1 already gives us $f \leq \frac{\pi}{2}$. We now strengthen it to $f < \theta_0$. First it is easy to see that $f \leq \theta_0$. Otherwise there is a point $x_0 > 0$ such that $f(x_0) > \theta_0$. Thus, we can find two points x_1, x_2, with $0 \leq x_1 < x_0 < x_2$, such that $f(x_1) = f(x_2)$ and $f(x) \geq f(x_1)$ for all $x \in (x_1, x_2)$. Modify f to $\tilde{f}$ by setting $\tilde{f}(x) = f(x_1)$, $x \in (x_1, x_2)$; $\tilde{f} = f$, elsewhere. Then $(a, \tilde{f}, h, g) \in \mathcal{C}$ and $L(a, \tilde{f}, h, g) < L(a, f, h, g)$ because f cannot be constant-valued over (x_1, x_2) by virtue of the equation (2.17) and the energy density $\mathcal{E}_1$ defined in (2.11) increases for $f \in [0, \frac{\pi}{2}]$. This contradiction implies that $f \leq \theta_0$. Next, we prove that $f < \theta_0$. Otherwise, if $f(x_0) = \theta_0$ for some $x_0 > 0$, then x_0 is a maximum point of f such that $f'(x_0) = 0$ and $f''(x_0) \leq 0$. Inserting these results into (2.17), we arrive at a contradiction since $0 < \theta_0 < \frac{\pi}{2}$.

To see that f is non-decreasing, we assume otherwise that there are $0 < x_1 < x_2$ such that $f(x_1) > f(x_2)$. Since $f(0) = 0$, we see that f has a local maximum point x_0 below x_2, which is known to be false. Thus f is non-decreasing. To see that f is strictly increasing, we assume otherwise that there are $0 < x_1 < x_2$ such that $f(x_1) = f(x_2)$. Hence f is constant-valued over $[x_1, x_2]$ which is impossible.

The proof of the lemma is complete.

Lemma 5.2 For the solution (a,f,h,g), we have the asymptotic estimates

$$a(x)=\mathrm{O}\left(\mathrm{e}^{-\alpha(1-\varepsilon)x}\right),\quad g(x)=q+\mathrm{O}\left(x^{-1}\right),$$
$$f(x)=\theta_0+\mathrm{O}\left(x^{-1}\right),\quad h(x)=1+\mathrm{O}\left(x^{-1}\mathrm{e}^{-\beta(1-\varepsilon)x}\right) \tag{5.1}$$

as $x\to\infty$, where $\varepsilon\in(0,1)$ is arbitrarily small and

$$\alpha=\sqrt{1-q^2+\xi\sin^2\theta_0}\ , \tag{5.2}$$

$$\beta=\sqrt{\frac{3}{4}\lambda+1}\ . \tag{5.3}$$

Proof To obtain the asymptotic estimate of a, we introduce a comparison function η,

$$b(x)=C\mathrm{e}^{-\alpha(1-\varepsilon)x},\ x>0. \tag{5.4}$$

From (2.16) and the obtained asymptotic behavior of a,f,h,g, we see that there is a sufficiently large $x_\varepsilon>0$ so that

$$\begin{aligned}(a-b)''=&\,a\left(\frac{a^2-1}{x^2}+h^2-g^2+\xi\sin^2 f+\kappa\sin^2 f\left(f'^2+\frac{a^2\sin^2 f}{x^2}\right)\right)-\\&C\alpha^2(1-\varepsilon)^2b\\ \geq&\,a(1-q^2+\xi\sin^2\theta_0)(1-\varepsilon)^2-\alpha^2(1-\varepsilon)^2b\\ =&\,\alpha^2(1-\varepsilon)^2(a-b),\qquad x>x_\varepsilon.\end{aligned}$$

Take the coefficient C in (5.4) large enough to make $a(x_\varepsilon)-b(x_\varepsilon)<0$. Since $a-b$ vanishes at infinity, applying the maximum principle in the above inequality, we derive the bound $a<b$ for $x>x_\varepsilon$ as desired.

To get the estimate for g, from (3.8) we see that

$$g'(x)=\frac{1}{x^2}\int_0^x 2a^2(s)g(s)\,\mathrm{d}s,\quad x>0, \tag{5.5}$$

which leads to

$$q-g(x)=\int_x^\infty\frac{1}{s^2}\int_0^s 2a^2(s')g(s')\,\mathrm{d}s'\,\mathrm{d}s=\mathrm{O}(x^{-1}), \tag{5.6}$$

for $x>0$ large, since $a(x)$ vanishes exponentially fast at $x=\infty$.

Similarly, we obtain the asymptotic estimate of f.

Set $H=x(h-1)$, using the fact $h(\infty)=1$ and the estimate of a in (decay), we have

$$\begin{aligned}H''=&\,\frac{(x^2h')'}{2x}=\lambda H\frac{h(h+1)}{2}+h\frac{a^2}{x}\\ \leq&\,\frac{3}{4}\lambda hH+H\frac{a^2}{x^2}\leq\beta^2(1-\varepsilon)^2H,\qquad x>>1.\end{aligned}$$

Let $c(x)=C\mathrm{e}^{-\beta(1-\varepsilon)x}$ be a comparison function. Then there is a sufficiently large $x_\varepsilon>0$ so that

$$(H+\eta)''\leq\beta^2(1-\varepsilon)^2(H+c),\qquad x\geq x_\varepsilon. \tag{5.7}$$

For fixed x_ε, we take the constant C in c large enough to make $H(x_\varepsilon)+c(x_\varepsilon)>0$. From the finite-energy condition we get that there exist a sequence $\{x_j\}$: $x_j\to\infty$ $(j\to\infty)$, such that $H(x_j)\to 0$. Furthermore, we have $H(x_j)+c(x_j)\to 0$ $(j\to\infty)$. Applying the maximum principle in (5.7), we derive that $H+c>0$, i.e. $0<x(1-h)<c$ for $x\geq x_\varepsilon$. Then we get $h=1+\mathrm{O}(x^{-1}\mathrm{e}^{-\beta(1-\varepsilon)x})$ as $x\to\infty$. The proof of the lemma is complete.

Lemma 5.3 For the solution (a,f,h,g) with fixed $\theta_0\in(0,\pi/2)$, the electric charge

$$Q_e(q)=2\int_0^\infty a^2(x)g(x)\mathrm{d}x \tag{5.8}$$

enjoys the property $Q_e(q)\to 0$ as $q\to 0$.

Proof For fixed θ_0, since a vanishes exponentially fast at infinity uniformly with respect to $q\in(0,1)$ and $0<g(x)<q$ for all $x>0$, we see that we can apply the dominated convergence theorem to (5.8) to conclude that $Q_e\to 0$ as $q\to 0$.

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