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PC-rings and Almost Excellent Extensions

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Abstract: In this paper we consider that rings R are coherents and R is p-injective. We call such rings to be right PC-rings. The structure of these rings is examined and if S is PC-ring if and only if R is PC-ring, where S is an almost excsllent extension of R.

Key words: PC-rings; almost excellent extensions
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§1. Introduction

We recall that a study of rings with R coherent and R coflat was make by Robert.F. Damiano[2]. In the same paper he has provided a number of necessary and sufficient conditions for R.

In this paper we consider rings with R coherent and R p-injective. We call such rings to be right PC-rings. Some properties of such rings are given in section 2.

In section 3 we consider almost excellent extension of PC-rings. We proved that R is a right PC-ring if and only if S is right PC-rings, where S is an excellent extension of R (for example $S=Mn(R)$, the $n\times n$, matrix ring or $S=R*G$, the corssed product where G is a finite group such that $o(G)^{-1}\in R$).

In the final section we consider smash products. We show that if G is a finite group such that $o(G)^{-1}\in R$. Then the smash product $R\# G^{*}$ is a right PC-ring if and only if R is right PC-ring.

Throughout the paper, all rings have a unity and all modules are unitary.

§2. *PC*-rings

A right R-module M is called a coflat if for any finitely generalized right ideal I of R and

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any R-homomorphism $f: I\to M$ there exist an $m\in$ M such that $f(x)=mx(x\in \mathrm{I})$. A ring R is called a right FC-ring if R is right coherent and R is coflat. Examples of FC-rings are given in [2].

Analogously, we give the following definition.

Definition 2.1 A ring R is called a right (left) PC-ring if R is right (left) coherent ring and R_R ($_RR$) is a p-injective module. A ring R is an PC-ring if it is both left and right PC-ring.

It is clear that right (left) FC-rings are right (left) PC-rings. QF ring and FC-ring are PC-ring.

If R is a ring and $X\subseteq R$, then we denote the left and right annihilators of X, respectively by

$$Ann_r(X)=\{a\in \mathrm{R} \mid ax=0 \quad \forall x\in \mathrm{X}\}.$$

$$Ann_l(X)=\{a\in \mathrm{R} \mid xa=0 \quad \forall x\in \mathrm{X}\}.$$

Definition 2.2 Let $\mathscr{S}$ and $\mathscr{R}$ sets of left and right iedeals of R, respectively. We call R has the Double annihilator property for $\mathscr{S}$ and $\mathscr{R}$ if

$$I\in\mathscr{S}\Rightarrow Ann_r(I)\in\mathscr{R} \text{and } Ann_lAnn_r(I)=I.$$

$$I\in\mathscr{R}\Rightarrow Ann_l(I)\in\mathscr{R} \text{and } Ann_rAnn_l(I)=I.$$

Theorem 2.3 For a ring, the following statements are equivalent

(1) R is PC ring.

(2) R is coherent and every right (left) flat module is right (left) p-injective.

(3) R is coherent and has the double annihinlator property for principal left and right ideal.

Proof (1)$\Leftrightarrow$(2) Clear.

(1)$\Rightarrow$(3) Let R_R be p-injective and $a\in R$, $Ra\subseteq Ann_lAnn_r(a)$ and we have $xAnn_r(a)=0$ for $\forall x\in Ann_lAnn_r(a)$.

Let $f: aR\to R: ar\to xr$.

Obviously f is a right R-homomorphism, then there is a element $b\in aR$ such that $sr=f(ar)=bar$, $x=ba\in Ra$. Thus $Ann_lAnn_r(a)=Ra$.

Similarly, we have $Ann_rAnn_l(a)=aR$.

(3)$\Rightarrow$(1) Suppose $f: aR\to R$ and $Ann_lAnn_r(a)=Ra$, then we have $0=f(0)=f(ab)=f(a)b$ for $b\in Ann_r(a)$ and $f(a)\in Ann_lAnn_r(a)=Ra$. So we have y in R such that $f(a)=ya$ and $f(ar)=f(a)r=(ya)r=y(ar)$. Thus R_R is p-injective.

Example 2.4 Let R be an algebra over a field F with basis

$$\{1, e_0, e_1, e_2, \cdots, x_1, x_2, x_3, \cdots\}$$

where for all i, j

$$e_ie_j=\delta_{ij}e_j$$

$$e_ie_j=\delta_{i(j+1)}X_I$$

$$e_iX_j=\delta_{ij}X_j$$

$$X_i X_j = 0.$$

It is easy to see both that R is left coherent and that every R-homomorphism

$$f: {}_R I \to {}_R R$$

extends to one over R. Thus, R is left p-injective. However, R is not right p-injective since the homomorphism

$$x_1 R \to e_0 R, \qquad \text{via} \qquad x_r \to e_0 r,$$

can not be extended over R.

Thus R is left PC-ring but not a right PC-ring.

Example 2.5 Let R be the ring with underying group

$$R = Z \oplus Q/Z.$$

And with multiplication

$$(n_1, q_1)(n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1).$$

Then it is easy to see that R is a commucative coherent ring with Jacobson radical

$$J(R) = \{(n, q) \mid n = 0\}.$$

Then R is not Von Neumann regular ring but R is PC ring.

Recall that the left (right) singular ideal of R is

$$Z_l(R) = \{x \in \mathrm{R} \mid \mathrm{Ann}_l(x) \triangle {}_R R\}.$$

$$Z_r(R) = \{x \in \mathrm{R} \mid \mathrm{Ann}_r(x) \triangle \mathrm{R}_R\}.$$

In general, there are not equal and are unrelated to $J(R)$.

Proposition 2.6 If R is an PC ring, then $Z_r(R) = Z_l(R) = J(R)$.

Proof Let $z \in Z_r(R)$, for any $a \in R$ and $b = 1 - az$.

Let $F: bR \to R$; via $br \to R$.

Obviously, F is a right R-homomorphism and R is PC ring, so there is a element $y \in R$ such that $F(br) = ybr(\forall\, r \in R)$. Then we have $1 = F(b) = yb = y(1 - az)$ and $1 - az$ is quasi-regrlar element. So $z \in R$ and $Z_r(R) \subseteq J(R)$.

Next suppose $x \in J(R)$. We claim $x \in Z_r(R$. If not, then $x \in J - Z_r(R)$ and there exist a right ideal $I(\neq 0)$ such that $Ann(x) \oplus I$ is essential in R. Let $0 \neq b \in I$ and

$$F: xbR \to : xbr \to br.$$

Obviously f is a right R-homomorphsim and R is PC ring, then we have a element $y \in R$ such that $f(xbr) = yxbr$ for any $r \in R$. So $b = f(xb) = yxb$. As $x \in J$, we have $xb \in J$ and xb is a quasi-regular element. We claim $x = yxb = 0$, a contradiction.

Theorem 2.7 If R is PC ring with no nonzero nilpotent elements, then R is Von Nenmman regular.

Proof Let $x \in R$, then we chaim $Rx \cap Ann_l(xR) = 0$. For suppose $rss = 0$ then $xrxxrx = 0$. So since R has no nonzero. Nilpotent elements, $xrx = 0$. Likewise, $rxrx = 0$ implies $rx = 0$. So $Rx \cap Ann_l(xR) = 0$. R has the double annihilator property for principal left and right ideals,

$$Ann_r(Rx) + xR = R.$$

Let $1 = n + xs$ where $n \in Ann_r(Rx)$ and $0 \neq s \in R$. Then $x = nx + xsx$. But $nxnx = 0$ so $nx = 0$ and $xsx = x$.

Corollary 2.8 If nonsingular ring R is a commutative PC ring then R is Von Neumann regular.

Proposition 2.9[13] If nonsingular ring R is PC ring with the ascending chain conditon for special right annihilators, then R is Von neumann regualr ring.

Proposition 2.10 If R is PC ring and nonsingular ring then there is a unique largest two side ideal I that contains no nonzero niloptent elements. Moreover $Ann_l Ann_r(I) = Ann_r Ann_l(I) = I$.

Proof Let $I = \sum_{i \in A} I_i$ where $\{I \mid i \in A\}$ is the family of all two side ideals of that contains no nonzero nilpotent elements.

First we will show that $Ann_r Ann_l(I)$ contains no nonzero nilpotent elements. So suppose $0 \neq \in Ann_r Ann_l(I)$ such that $x^2 = 0$. If $R_x \cap I_i = 0$ for all $i \in A$. Then

$$I_i Rx \subseteq \mathrm{Rx} \cap \mathrm{I}_i = 0, \text{ for any } i \in \mathrm{A} \text{ and } RX \subseteq \mathrm{r(I)}.$$

So

$$RxRx \leqslant \mathrm{Ann}_l Ann_r(I) Ann_r(I) = 0.$$

And $RxRx = 0$. By Proposition 2.6, so $Rx = 0$ and $x = 0$. Thus a contradiction.

Claim: there is a $i_0 \in A$ such that $Rx \cap I_i \neq 0$ and $0 \neq a \in Rx \cap I_i$, then $a^2 \neq 0$. Suppose $b \in Ann_l(a^2)$, then $aba \in Ra$ and $aba = 0$, $(ba)^2 = b(aba) = 0$, so $ba = 0$. Thus $Ann_l(a) = Ann_l(a^2)$. By Theorem 2.3 we have

$$aR = Ann_r Ann_l(a) = Ann_r Ann_l(a^2) = a^2 R.$$

Then there is a element $c \in R$ such that $a^2 = a^2 c$ and $(a - aca)^2 = 0$, $a - aca \in Ra$, so $a = aca$. Let $e = ca$ then $e^2 = (ca)^2 = c(aca) = ca \in Ra \subseteq Rx$, then $e = dx$ and $(xe)^2 = xdx^2 dx = 0$, $xe \in Ra$. Thus $xe = 0$, $e = e^2 = dxe = 0$, $a = aca = ae = 0$, a contradiction.

§3. Almost Execllent Extensions

Suppose that R is a subring of the ring S, P and S have the same identity. The ring S is said to be an excellent extension of R if

(A) S is a free normalizing extension of R with a basis that includes 1; that is; there exsits a finite set $\{a_1, \cdots, a_n\} \subseteq S$ such that $a_= 1$, $S = a_1 R + \cdots + a_n R$ and $a_i R = R a_i$ for all $i = 1 - n$ and S is free with asis $\{a_1, \cdots, a_n\}$ as both a right and left R-module and

(B) S is R-projective, that is, if Ns is a submodule of Ms, then $N_R \mid M_R$ implies $N_s \mid M_s$.

Excellent extensions were introduced by Passman [8] and named by Bonami [9]. Example include finite matrix rings [8] and crossed prodcts $R * G$, where G is a finite group $o(G)^{-1} \in R$[9]. Further examples are given in [8] and [10], some authers consider various properties which shared by R and S when S is an excellent extension of R.

The ring S is said to almost excellent extension of R if the conditions (B) and (C) are satisfied.

(C) S is a finite normalizing extension of R such that S_R is projective R-module and S_R is flat.

Obviously excellent extension is an almost excellent extension.

A module M is p-injective if for any principal right ideal I of R, any R-homomorphism $g: I \to M$. Can be extended to a p-homomorphism $R \to M$.

Lemma 3.1 Let S be alost excellent extension of R and Ms p-injective then M_R is a p-injective.

Proof Suppose that U is a principal ideal of R and $I = bR$ and R-homomorphism $f: I \to M$. Let $J = Ia_1 + \cdots Ia_n$, then J is a principal right ideal of S by $J = bS$. Let $F: J \to M$ such that $F(\sum_{i=1}^{n} x_i a_i) = \sum \mathrm{f}(\mathrm{x}_i) a_i$. Obviously, F is a S-homomorphism, then it exists a S-homomorphism $G: S \to Ms$ suth that $G|_J = F$. Thus $G|_R: R \to M$ is a extension of f.

Lemma 3.2[12] Let S be an almost excellent extension of R, then S right coherent if and only if R is a right coherent.

Theorem 3.3 Let S be an almost excellent extension of R, then R is right PC-ring if and only if R is a right PC-ring.

Proof S is a right PC-ring if and only if S is a right coherent ring and S_s p-injective if and only if R is a right coherent ring and S_R p-injective by Lemma 3.1 and Lemma 3.2. Obviously, $R_R | S_R^T$ and $S_R | R_R^T$ by S almost excellent extension of R, we see that S_R is p-injective if and only if R_R is p-injective. Thus S is a right PC-ring if and only if R is a right PC-ring.

Remark 3.4 By analogy with the proof of Theorem we cah show that if S is an almost excellent extension of R, then S is a right FC-ring if and only if R is a right FC-ring. Let S be an almost excellent extension of R.

Corollary 3.5 Let S be an almost excellent extension of R, the following statements are equivalent

(1) R is right PC-ring.

(2) The matrix ring $Mn(R)$ is a right PC-ring.

(3) The crossed product $R * G$ is a right PC-ring (G finite group ang $o(G)^{-1} \in R$).

§ 4. Smash Products

Let R be graded by a finite group G. The sash product $R \# G^*$ is a free right and left R-module with basis $\{pa | a \in G\}$ and multiplication determined by

$$(rp_a)(sp_b) = rs_{ab}{}^{-1} p_b$$

where $s_{ab}{}^{-1}$ is the ab^{-1} component of s.

Theorem 4.1 Let R be graded by a finite group G and $o(G)^{-1} \in R$. Then the smash product $R \# G^*$ is a right PC-ring if and only if R is right PC-ring.

Proof The group G acts as automorphisms on $R\# G^{*}$ with $g(rp_a)=rp_{ag}$, so we may form the skew group ring $(R\# G^{*})*G$. The duality Theorem for coactions[14 Theorem3.5] asserts that $(R\# G^{*})*G\cong Mn(R)$, the ring of $n\times n$ matrices over R, where $n=o(G)$. Thus the result follows from Corollary 3.5.

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PC-环与几乎优越扩张

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摘 要：本文我们讨论了这种环 R 的结构，R 是凝聚环且 R 作为 R-模是 P-内射，我们称此环为 PC-环。并证明了在几乎优越扩张下的不变性。

关键词：PC-环；几乎优越扩张