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Existence and Nonexistence of the Global Solution on the Quasilinear Parabolic Equation

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Abstract: The paper studies the existence, the exponential decay and the nonexistence of global solution for a class of quasilinear parabolic equations.
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§1. Introduction

In this paper we are concerned with the existence, asymptotic and nonexistence of global solutions for the following nonlinear parabolic equation

$$|u_t|^{m-2}u_t = \Delta u + |u|^{p-2}u, \qquad (t,x) \in J \times \Omega, \tag{1.1}$$

$$u(t,x) = 0, \qquad (t,x) \in J \times \partial\Omega, \tag{1.2}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $J = [0,\infty)$ and $\Omega \subset R^n$($n \geq 1$ is a natural number) is a bounded domain with smooth boundary $\partial\Omega$, $m,p > 1$ are positive integer. Some special cases of (1.1) have already been treated in the literatures. For example, when $m = 2$, the initial-boundary value problem (1.1)—(1.3) has been studied by many authors, see [1—5] and their references. Pucci and Serrin [6—8] discuss the stability and blow-up of the following equation

$$A(t)|u_t|^{m-2}u_t - \Delta u + f(x,u) = 0, \tag{1.4}$$

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where $f(u)$ represents a restoring force(i.e., $f(u)u > 0$)). To our knowledge, there are no any other result for the equation (1.4) with source terms (i.e., $f(u)u < 0$)), so that a study of this problem seems to be worthwhile. On the other hand, the situation for second order hyperbolic systems has received much attention. Recently, various extensions of the asymptotic stability and nonexistence of solution have been given for the case of second order hyperbolic systems. For example, the following equation

$$(|u_t|^{l-2}u_t)_t - \nabla(|\nabla u|^{q-2}\nabla u) + |u_t|^{m-2}u_t = |u|^{p-2}u$$

and the abstract equation of the form

$$(P(u_t))_t + A(u) + Q(t, u_t) = F(u)$$

have been many results, see [7—11, 18, 19, 20] and their references.

The purpose of this paper is to establish the existence, asymptotic behavior and nonexistence of global solution for problem (1.1)—(1.3). Our treatment is based principally on the references [4, 6, 7, 8]. We will get the global existence by Galerkin method [4]. In particular, we will improve and extend the result of asymptotic behave in [6, 7], that is, we can get exponential decay of energy under the same condition as [6, 7]. Our method is different from that of reducing (3.13) from (3.1) in [14] and [15] . Furthermore, Todorova [14] and Aassila [15] get only the polynomial decay. We get the exponential decay, we can also generalize these results by replacing equation (1.1) with general forms

$$A(t)|u_t|^{m-2}u_t = \Delta u + f(x, u).$$

This paper is organized as follows. In section 2 we give the global existence and energy decay and in section 3, we shall establish the blow-up result.

§2. Global Existence and Asymptotic Behavior

We shall use the notations employed in the book of Lions [12]. We always assume that $2 < p \leq \infty$ if $n = 1, 2$ or $2 < p \leq \frac{2n}{n-2}$ if $n \geq 3$, and $2 \leq m < p$. Following the ideals of the "potential well theory "introduced by Payne and Sattinger [13] and Lions [12], we define the functional

$$E(t) = E(u(t)) = \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|_p^p \tag{2.1}$$

for $u \in H_0^1(\Omega)$. In this case the "potential depth" given by

$$d = \inf\{\sup_{\lambda>0} E(\lambda u); u \in H_0^1 \setminus \{0\}\}$$

become positive. Then we are able to define stable and unstable sets. Denote

$$W = \{u \in H_0^1 | K(u) > 0,\ E(u) < d\} \bigcup \{0\},$$

where

$$K(u) = \|\nabla u\|^2 - \|u\|_p^p.$$

Lemma 2.1 Suppose that $u_0 \in W$ and $u(t,x)$ be a local solution of problem (1.1)—(1.3) on $[0, t_{\max})$. If $E(0) = E(u_0) < d$ then $u(.,t)$ remains inside the set W for any $t \in [0, t_{\max})$.

Lemma 2.2 Let $u_0 \in W$ and $u(x,t)$ be a local solution of problem (1.1)—(1.3) on $[0, t_{\max})$, then the inequality

$$E(u(t)) \geq \frac{p-2}{2p} \| \nabla u \|^2 \tag{2.2}$$

is fulfilled for $t \in [0, t_{\max})$.

The proof is similar to that of Lemma 2.2 and 3.1 in [14], so we omit it.

Theorem 2.3 Let $u_0 \in W$, $2 < p < \infty$ if $n \leq 2$ and $2 < p < \frac{2n}{n-2}$ if $n > 2$, then the initial boundary value problem (1.1)—(1.3) has a solution $u \in W$, such that

$$u \in L^\infty(0,T;H_0^1),\ u_t \in L^m(0,T;L^m(\Omega)).$$

The proof is similar to [4], by employing Galerkin's method, so we omit it.

In order to get asymptotic stability of the solution of the problem (1.1)—(1.3), we introduce the another representative form of W [17—19]:

$$W = \{v \in H_0^1 | 0 \leq \| \nabla v \| < \lambda_1,\ 0 \leq E(t) = \frac{1}{2} \| \nabla v \|^2 - \frac{1}{p} \|v\|_p^p < d\},$$

where $\lambda_1 = (\frac{1}{pC_0^p})^{\frac{1}{p-2}}$, $d = \lambda_1^2(\frac{1}{2} - \frac{1}{p})$ and C_0 is the embedding constant. Then we have the following results.

Lemma 2.4 Let u be a solution of the problem (1.1)—(1.3). If $u_0 \in W$, then we have

$$\| \nabla u \|^2 \geq 2 \|u\|_p^p. \tag{2.3}$$

Proof By the definition of $E(t)$ and embedding theorem, we have

$$\begin{aligned} E(u(t)) &= \frac{1}{2} \| \nabla u \|^2 - \frac{1}{p} \|u\|_p^p \\ &\geq \frac{1}{2} \| \nabla u \|^2 - \frac{C_0^p}{p} \| | \nabla u \|^p \\ &\geq \frac{1}{2} \| \nabla u \|^2 - C_0^p \| \nabla u \|^p \\ &\geq G(\| \nabla u \|), \end{aligned} \tag{2.4}$$

where $G(\lambda) = \frac{1}{2}\lambda^2 - C_0^p \lambda^p$. It is easy to see that $G(\lambda)$ attains its maximum d for $\lambda = \lambda_1$, $G(\lambda)$ is strictly decreasing for $\lambda > \lambda_1$ and $G(\lambda) \to -\infty$ as $\lambda \to \infty$. Since $E(u(t)) < d$ for $t \in R^+$, by Lemma 2.1, we have $\| \nabla u \| < \lambda_1$ for $t \in R^+$, and then we get $G(\| \nabla u \|) \geq 0$ for $0 \leq \| \nabla u \| < \lambda_1$. We notice that

$$\| \nabla u \|^2 - \|u\|_p^p = \frac{1}{2} \| \nabla u \|^2 + (\frac{1}{2} \| \nabla u \|^2 - \|u\|_p^p) \geq \frac{1}{2} \| \nabla u \|^2 + G(\| \nabla u \|).$$

Hence (2.3) holds since $0 \leq \| \nabla u \| < \lambda_1$ for $t \in R^+$ and $G(\| \nabla u \|) > 0$ for $0 \leq \| \nabla u \| < \lambda_1$.

Lemma 2.5([16]) Let $\Phi(t)$ be a bounded nonnegative function on R^+ satisfies

$$\sup_{s\in[t,t+1]} \Phi(s)^{1+\alpha} \le K_0(\Phi(t)-\Phi(t+1)), \quad t\in R^+,$$

where $\alpha\ge 0, K_0$ is a positive constant. Then we have

(i) If $\alpha>0$, then $\Phi(t)\le C_0 t^{-\frac{1}{\alpha}},\ t>0$;

(ii) If $\alpha=0$, then we have $\Phi(t)\le C_1 e^{-C_2 t},\ t>0$.

In the above, $C_i (i=1,2)$ are constants depending $\Phi(0)$ and other known constants.

Theorem 2.6 Let $2\le m\le p,\ u_0\in W$ and u be a global solution of problem (1.1)—(1.3), then the solution of the problem has the following decay property

$$\|\nabla u(t)\|\le Ce^{-\lambda t},$$

where λ, C are positive constants.

Proof First, by (1.1) we have

$$\int_t^{t+1}\|u_t(s)\|_m^m \mathrm{d}s = E(u(t))-E(u(t+1)). \tag{2.5}$$

Multiplying Equation (1.1) by u and integrating over $\Omega\times[t,t+1]$, we have

$$\begin{aligned}
&\int_t^{t+1}(\|\nabla u(s)\|^2-\|u(s)\|_p^p)\mathrm{d}s\\
&=-\int_t^{t+1}\int_\Omega |u_t(s)|^{m-2}u_t(s)u(s)\mathrm{d}x\mathrm{d}s\\
&\le\int_t^{t+1}\|u_t(s)\|_m^{m-1}\|u(s)\|_m\mathrm{d}s\\
&\le\Big(\int_t^{t+1}\|u_t(s)\|_m^m\mathrm{d}s\Big)^{\frac{m-1}{m}}\Big(\int_t^{t+1}\|u(s)\|_m^m\mathrm{d}s\Big)^{\frac{1}{m}}.
\end{aligned} \tag{2.6}$$

Now, from the convexity of the function u^y/y in y for $u\ge 0$ and for $y\ge 0$, we obtain

$$\frac{\|u\|_m^m}{m}\le C_1\frac{\|u\|_2^2}{2}+C_2\frac{\|u\|_p^p}{p},$$

since $2\le m\le p$, where C_1, C_2 are positive constants. So we have, by Poincare inequality and Lemma 2.4

$$\begin{aligned}
\|u(s)\|_m^m &\le C_1\|u(s)\|_2^2+C_2\|u(s)\|_p^p\\
&\le C_3\|\nabla u(s)\|^2\\
&\le C_4E(u(s)).
\end{aligned} \tag{2.7}$$

Then, from (2.6), together with (2.7) and Young inequality, we obtain

$$\begin{aligned}
&\int_t^{t+1}(\|\nabla u(s)\|^2-\|u(s)\|_p^p)\mathrm{d}s\\
&\le\Big(\int_t^{t+1}\|u_t(s)\|_m^m\mathrm{d}s\Big)^{\frac{m-1}{m}}\Big(\int_t^{t+1}C_4E(u(s))\mathrm{d}s\Big)^{\frac{1}{m}}\\
&\le C(\epsilon)\int_t^{t+1}\|u_t(s)\|_m^m\mathrm{d}s+\epsilon\int_t^{t+1}E(u(s))\mathrm{d}s
\end{aligned} \tag{2.8}$$

for any $\epsilon > 0$. On the other hand, by using (2.2) and (2.3), we have

$$\begin{aligned} pE(u(s)) \leq & \frac{p}{2}(\|\nabla u\|^2 - \|u\|_p^p) + \frac{p-2}{4}\|\nabla u\|^2 \\ \leq & \frac{p}{2}(\|\nabla u\|^2 - \|u\|_p^p) + \frac{p}{2}E(u(s)). \end{aligned}$$

Hence

$$E(u(s)) \leq \|\nabla u\|^2 - \|u\|_p^p. \tag{2.9}$$

Now, choosing ϵ sufficiently small, using (2.9) together with (2.5), and we rewrite (2.8) in the form

$$\begin{aligned} \int_t^{t+1} E(u(s))\mathrm{d}s \leq & C_4 \int_t^{t+1} \|u_t(s)\|_m^m \mathrm{d}s \\ \leq & C_4(E(u(t)) - E(u(t+1))). \end{aligned} \tag{2.10}$$

Since $E(u(t)) > 0$, we can choose $t_0 \in [t, t+1]$ such that

$$E(u(t_0)) \leq C_5 \int_t^{t+1} E(u(s))\mathrm{d}s. \tag{2.11}$$

By means of (2.5), we know $E(u(t+1)) \leq E(u(t_0))$, and then

$$\begin{aligned} E(u(t)) = & E(u(t+1)) + \int_t^{t+1} \|u_t(s)\|_m^m \mathrm{d}s \\ \leq & E(u(t_0)) + \int_t^{t+1} \|u_t(s)\|_m^m \mathrm{d}s. \end{aligned} \tag{2.12}$$

By means of (2.10)—(2.12), we arrive at

$$\sup_{t \leq s \leq t+1} E(u(s)) \leq C_6(E(u(t)) - E(u(t+1))).$$

From Lemma 2.5, we have

$$E(u(t)) \leq C_7 e^{-C_8 t}.$$

Therefore, from Lemma 2.2

$$\|\nabla u\| \leq C_8 e^{-C_9 t}.$$

§3. The Problem of Blow-up

The idea follows from [20], we give the proof for continence.

Theorem 3.1 Suppose that $1 < m < p, u_0 \in H_0^1$. Let $u(x,t)$ be a local solution of problem (1.1)—(1.3) on $[0, T_{\max})$. Then no solution of (1.1)—(1.3) can exist on $J = [0, \infty)$ when $E(0) < 0$.

Proof Assume for contradiction that there is a solution of (1.1)—(1.3) on J. Define

$$H(t) = \int_0^t \|u_t(s)\|_m^m \mathrm{d}s - E(u_0). \tag{3.1}$$

Hence

$$||u||_p^p = p(\frac{1}{2}||\nabla u||^2 + H(t)) \geq pH(t) \geq -pE(u_0) > 0. \tag{3.2}$$

Multiplying equation (1.1) by u and integrating over Ω, we have

$$\begin{aligned} 0 =& ||u||_p^p - ||\nabla u||^2 - (|u_t|^{m-2}u_t, u) \\ =& \frac{p+2}{p}||u||_p^p - 2E(u(t)) - (|u_t|^{m-2}u_t, u) \\ \geq& \frac{p+2}{p}||u||_p^p - 2E(u(t)) - ||u_t||^{m-1}_{\frac{p(m-1)}{p-1}}||u||_p. \end{aligned} \tag{3.3}$$

Recalling $E(u(t)) < E(u_0) < 0$ on J, we see from (3.2) and (3.3) that

$$||u_t||^{m-1}_{\frac{p(m-1)}{p-1}} \geq \frac{p+2}{p}||u||_p^{p-1} \geq \frac{p-2}{p}(pH(t))^{\frac{p-1}{p}}. \tag{3.4}$$

Since $1 < m < p$, we have $m \geq \frac{p(m-1)}{p-1}$, this combine embedding theorem and (3.4), we have

$$H'(t) = ||u_t||_m^m \geq C||u_t||^m_{\frac{p(m-1)}{p-1}} \geq C(H(t))^{\frac{p-1}{p}\cdot\frac{m}{m-1}}.$$

Now, since $1 < m < p$, by assumption,

$$\frac{p-1}{p}\cdot\frac{m}{m-1} - 1 = \frac{(p-1)m-(m-1)p}{(m-1)p} = \frac{p-m}{(m-1)p} > 0,$$

so we can write $\frac{(p-1)m}{(m-1)p} = 1+\theta,\ \theta > 0$, then

$$\frac{H'}{H^{1+\theta}} > C.$$

By integration setting $H_0 = H(0) = -E(0) > 0$,

$$\frac{1}{\theta H_0^\theta} \geq \frac{1}{\theta[H(t)]^\theta} + Ct.$$

This is impossible, since the left hand side is finite and the right hand side goes to ∞ as $t \to \infty$.

[References]

[1] BALL J. Remarks on blow-up and nonexistence theorems for nonlinear evolution equation[J]. Quart J Math Oxford Ser, 1977, 28(2): 473-486.

[2] FUJITA H. On the blowing up of solution of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$[J]. J Fac Sci Univ Tokyo Sect IA Math, 1966, 13: 109-124.

[3] LEVINE H A. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$[J]. Arch Rational Mech Anal, 1973, 51: 371-386.

[4] TSUTSUM M. Existence and nonexistence of global solutions for nonlinear parabolic equations[J]. Pull Res Inst Math Sci, 1972, 8: 211-229.

[5] SAMARSKII A, GALAKTIONOV V, KURDYUMOV S, MIKHAILOV A. Blow-up in Quasilinear Parabolic Equation, de Gruyter Expositions in Mathematics[M]. Berlin: Walter de Gruyter and Co, 1995.

[6] PUCCI P, SERRIN J. Asymptotic Stability for Nonlinear Parabolic Systems[C]. Kluwer: Dordrecht, 1996, 66-74.

[7] PUCCI P, SERRIN J. Asymptotic stability for nonautonomous dissipative wave systems[J]. Communications on Pure and Applied Mathematics, 1996, 49: 177-216.

[8] PUCCI P, SERRIN J. Stability and Blow-up for Dissipative Evolution Equation[C]. In: Reaction-diffusion Systems(G Carirto, E Mitidieri, eds), Marcel Dekker, 1997, 299-316.

[9] BOCUTO A, VITILLARO E. Asymptotic stability for abstract evolution equation and application to partial differential systems[J]. Rend Circ Mat Palermo, 1998, 47: 25-48.

[10] SERRIN J. G.Todorova, E.Vitillaro, Existence for a nonlinear wave equation with damping and source terms[J]. Diff Integ Equa, 2003, 16: 13-50.

[11] VITILLARO E. Some new results on global nonexistence and blow-up for evolution problems with positive initial energy[J]. Rend Istit Mal Univ Trieste, 2000, 31: 245-275.

[12] LIONS J L. Quelques Methods de Resolution des Problem aux Limites Non linears[M]. Dunod: Paris, 1969.

[13] SATTINGER D H. On global solutions of nonlinear hyperbolic equations[J]. Arch Rational Mech Anal, 1968, 30: 128-172.

[14] TODOROVA G. Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms[J]. J Math Anal Appl, 1997, 239: 213-226.

[15] AASSILA M. Global existence and global nonexistence of solutions to a wave equation with nonlinear damping and source terms[J]. Asymptotic Analysis, 2002, 30: 301-311.

[16] NAKAO M. Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative[J]. J Math Anal Appl, 1977, 58: 336-343.

[17] VITILLARO E. Some new results on global nonexistence and blow-up for evolution problem with positive initial energy[J]. Rend Istit Math Univ Trieste, 2000, 31(2): 245-275.

[18] ZHANG Hong-wei, CHEN Guo-wang. Asymptotic stability for a nonlinear evolution equation[J]. Comment Math Univ Carolinac, 2004, 45(1): 101-107.

[19] ZHANG Hong-wei, HU Qing-ying. Existence of global weak solution and stability of a class nonlinear evolution equation[J]. Acta Mathematica Scientia, 2004, 24A(3): 329-336.

[20] LEVINE H A, PARK S, SERRIN J. Global existence and nonexistence theorem for quasilinear evolution equations of formally parabolic type[J]. J Diff Equat, 1998, 142: 212-229.