Chin. Quart. J. of Math.
2011, 26 (1): 93—99

Factorisable Quasi-adequate Semigroups

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Abstract: In this paper, we investigate a class of factorisable IC quasi-adequate semigroups, so-called, factorisable IC quasi-adequate semigroups of type-(H, I). Some characterizations of factorisable IC quasi-adequate semigroups of type-(H, I) are obtained. In particular, we prove that any IC quasi-adequate semigroup has a factorisable IC quasi-adequate subsemigroups of type-(H, I) and a band of cancellative monoids.

Key words: factorisable semigroup; IC quasi-adequate semigroup; factorisable abundant semigroups of type-(H, I)

2000 MR Subject Classification: 20M10

CLC number: O152.7 **Document code**: A

Article ID: 1002–0462 (2011) 01–0093–07

§1. Introduction and Preliminaries

The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are generalization of Green's relations $\mathcal{L}$ and $\mathcal{R}$: elements a and b of a semigroup S are related by $\mathcal{L}^*$ [resp $\mathcal{R}^*$] in S if and only if they are related by $\mathcal{L}$ [resp $\mathcal{R}$] in some oversemigroup of S. A semigroup S is called abundant if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of S contains at least one idempotent. An abundant semigroup is termed an quasi-adequate semigroup if its idempotents form a subsemigroup, that is, a band. Regular semigroups are abundant semigroups while orthodox semigroups are quasi-adequate semigroups. Quasi-adequate semigroups have attracted due attractions. There are many works on this aspect (see, [2], [4], [7], [9-11], [14], [15], [19], [22] and others).

A semigroup S is said to be factorisable if it can be written as the set product AB of proper subsemigroups A and B. In this case, we call the pair (A, B) a factorization of S.

Received date: 2007-04-23

Foundation item: Supported by the NSF of Jiangxi Province(0511037)

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Tolo[21] researched factorizble completely regular semigroups. As Catino[1], we call a factorisable semigroup S with factorization (A,B) left univocal if S satisfies the following condition:

$$(\forall a,a'\in A;b,b'\in B)ab=a'b'\Rightarrow a=a'.$$

Dually, we can define right univocal factorisable semigroups. And a factorisable semigroup with factorization (A,B) is called univocal if it is both left univocal and right univocal. In [20], the authors and Shum investigated left univocal factorisable rpp semigroups.

The concept of orders, in particular, of various kinds of natural partial orders, are one of important concepts in the theory of semigroups. M V Lawson[18] introduced the following partial orders on abundant semigroups, which are probed by authors(see [10], [19], [13] and others). These orders can be equivalently defined by:

$$\begin{aligned}
a\leq_r b &\iff \text{for all [some] idempotent } b^\dagger\in R_b^*, \text{ there exists an idempotent}\\
&\qquad a^\dagger\in R_a^* \text{ such that } a^\dagger\omega b^\dagger \text{ and } a=a^\dagger b.\\
a\leq_\ell b &\iff \text{for all [some] idempotent } b^*\in L_b^*, \text{ there exists an idempotent}\\
&\qquad a^*\in L_a^* \text{ such that } a^*\omega b^* \text{ and } a=ba^*.\\
a\leq b &\iff \text{for some } e,f\in E(S),\ a=eb=bf.
\end{aligned}$$

An abundant semigroup S is called F-abundant if each σ-class contains a greatest element with respect to $\leq$, where σ is the smallest cancellative congruence on S. Guo[10] obtained many characterizations of such semigroups and in particular, established the structure of strongly F-abundant semigroups. Zhang-Chen[22] investigated u-IC quasi-adequate semigroups, which are special F-abundant semigroups. Recently, Ni-Chen-Guo[19] gave a construction method of general F-abundant semigroups. It is worth to pointing out that the concept of F-abundant semigroups is generalized to the range of rpp semigroups(see [12] and [17]).

In this paper we shall consider a kind of factorisable quasi-adequate semigroups. Many characterizations of this kind of factorisable semigroups are obtained.

Throughout this paper we use the notions and terminologies of Fountain[8] and Howie[16]. Now, we provide some known results repeatedly used without mentions in the sequel.

Lemma 1.1[8] Let S be a semigroup and $a,b\in S$. Then the following statements are equivalent:

(1) $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$.

(2) For all $x,y\in S^1$, $ax=ay$ $[xa=ya]$ if and only if $bx=by$ $[xb=yb]$.

As an easy but useful consequence of Lemma 1.1, we have

Lemma 1.2[8] Let S be a semigroup and $e^2=e,a\in S$. Then the following statements are equivalent:

(1) $a\mathcal{L}^*e$ $[a\mathcal{R}^*e]$.

(2) $ae=a$ $[ea=a]$ and for all $x,y\in S^1$, $ax=ay$ $[xa=ya]$ implies that $ex=ey$ $[xe=ye]$.

Evidently, $\mathcal{L}^*$ is a right congruence while $\mathcal{R}^*$ is a left congruence. In general, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. And, when a and b are regular elements, $a\mathcal{L}b$ [$a\mathcal{R}b$] if and only if $a\mathcal{L}^*b$ [$a\mathcal{R}^*b$]. For convenience, we denote by $E(S)$ the set of idempotents of S, and by K_a the $\mathcal{K}$-class of S containing a, where $\mathcal{K}$ is one of $\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*, \mathcal{D}^*$ and $\mathcal{J}^*$. Also, we use $a^\dagger$ [a^*] to denote the idempotents in $R_a^* \bigcap E(S)$ [$L_a^* \bigcap E(S)$].

An abundant semigroup S is called idempotent-connected, for short, IC, when for all $a \in S$ and for some $a^\dagger, a^*$, there exists a bijection θ: $\langle a^\dagger \rangle \to \langle a^* \rangle$ such that $xa = a(x\theta)$ for all $x \in \langle a^\dagger \rangle$, where $\langle e \rangle (e \in E(S))$ is the subsemigroup of S generated by the set of idempotents of eSe. Type-A semigroups are just IC adequate semigroups.

Lemma 1.3[5] Let S be an abundant semigroup. Then the following statements are equivalent:

(1) S is IC.

(2) $\leq_r = \leq = \leq_\ell$.

(3) For each $a \in S$ the following two conditions hold:

(i) For each $e \in \omega(a^\dagger)$ there exists $f \in \omega(a^*)$ [resp. $f \in S$] such that $ea = af$;

(ii) For each $g \in \omega(a^*)$ there exists $h \in \omega(a^\dagger)$ [resp. $h \in S$] such that $ag = ha$.
Where ω is the natural order on $E(S)$ and for $e \in E(S)$, $\omega(e) = \{f \in E(S) : f\omega e\}$.

The following result is used in the sequel.

Lemma 1.4[10] Let S be an IC quasi-adequate semigroup. Then the relation

$$\{(x, y) \in S \times S \mid exf = eyf \text{ fome some } e, f \in E(S)\}$$

is the smallest cancellative monoid congurence on S.

§2. Definitions and Characterizations

Definition 2.1 A factorisable monoid S with a factorization (A, B) is called type-(H, I) [type-(I, H)] if

(1) A [B] is a cancellative monoid contained in the $\mathcal{H}^*$-class of S containing 1, where 1 is the identity of S.

(2) $B \subseteq E(S)$ [$A \subseteq E(S)$].

In this case, (A, B) is called an (H, I)-factorizations [(I, H)-factorizations] of S.

The following proposition gives some properties of factorisable IC quasi-adequate semigroups of type-(H, I).

Proposition 2.2 Let S be a factorisable IC quasi-adequate monoid and (A, B) an (H, I)-factorization of S. Then

(1) $A = H_1^*$.

(2) $S = H_1^*E(S)$.

(3) $(E(S), A)$ is an (I, H)-factorization of S.

(4) If, in addition, (A, B) is a left [resp right] univocal factorization of S, then S is an F-abundant semigroup.

Proof (1) Obviously, $A \subseteq H_1^*$. Now let and $h = h'f \in H_1^*$ with $h' \in A$ and $f \in E$. Since $h'\mathcal{L}^*1$ and $\mathcal{L}^*$ is a right congruence, we have $h = h'f\mathcal{L}^*1f = f$ and $f\mathcal{L}^*1$ since $h\mathcal{H}^*1$. So, for some $x \in S$, $1 = xf = xff = 1f = f$. This shows that $h = h' \in A$, whence $H_1^* \subseteq A$. Therefore $A = H_1^*$.

(2) By (1), $S = AB \subseteq H_1^*E(S) \subseteq S$ and $S = H_1^*E(S)$.

(3) It suffices to verify that for any $s \in S$, there exist $h_s \in H_1^*$ and $e_s \in E(S)$ such that $s = e_s h_s$. Now let $s = hf$ with $h \in H_1^*$ and $f \in E(S)$. By the proof of (1), $s\mathcal{L}^*f$. Hence $s \leq_\ell h$ since $h\mathcal{H}^*1$ and $f \leq 1$. Now, by Lemma 1.3, $s \leq h$ and $s = kh$ for some $k \in E(S)$, as required.

(4) Without loss of generality, we assume that (A, B) is a left univocal factorization of S. By the proof of (3), for all $s \in S$, there exist $h_s \in H_1^*$ and $e_s \in E(S)$ such that $s = h_s e_s$ and $s \leq h_s$. Now, by Lemma 1.4, if $x\sigma y$(σ is the smallest cancellative monoid conguence on S), then $h_x e = h_y f$ for some $e, f \in E(S)$(since S is IC) and $h_x, h_y \in H_1^*$, hence $h_x = h_y$ since S is left univocal. This shows that for any $y \in \sigma_x$, $y = h_x f_y$ for some $f_y \in E(S)$, hence $y \leq h_x$. Consider that $h_x \in \sigma_x$. Thus h_x is the greatest element of σ_x. Therefore each σ-class of S contains a greatest element. That is, S is an F-abundant semigroup.

Theorem 2.3 Let S be an IC quasi-adequate monoid with identity 1. Then S is a factorisable semigroup of type-(H, I) if and only if for all $s \in S$, there exists $h \in H_1^*$ such that $s \leq h$.

Proof ($\Rightarrow$) It follows from the proof of Proposition 2.2(3).

($\Leftarrow$) It is straight from the definition of $\leq$.

Let S be a monoid and $a \in S$. Put

$$A^*(a) = \{b \in S : a = ebf \text{ for some } e, f \in E(S) \text{ with } e\mathcal{R}^*a, f\mathcal{L}^*a\}.$$

In what follows, we denote $U^*(a) = A^*(a) \cap H_1^*$.

Theorem 2.4 Let S be an IC quasi-adequate semigroup with identity 1. Then S is a factorisable semigroup of type-(H, I) if and only if $U^*(a) \neq \emptyset$, for all $a \in S$.

Proof ($\Rightarrow$) Assume that S is (H, I)-factorisable. Then, by Proposition 2.2, $S = H_1^*E(S)$. That is, for all $a \in S$, there exist $b \in H_1^*, f \in E(S)$ such that $a = bf$. Consider that $\mathcal{L}^*$ is a right congruence and that $b\mathcal{L}^*1$, we have $a = bf\mathcal{L}^*1f = f$. Pick $e \in E(S) \cap R_a^*$, we have $a = ea = ebf$, so that $b \in A^*(a) \cap H_1^*$. Thus $U^*(a) \neq \emptyset$.

($\Leftarrow$) If $U^*(a) \neq \emptyset$, then for any $a \in S$, there exists $b \in H_1^*$ such that $a = ebf$, where $e, f \in E(S)$ with $e\mathcal{R}^*a, f\mathcal{L}^*a$. Note that 1 is the greatest idempotent in S. We observe $eb \leq_r b$, hence by Lemma 1.3, $eb \leq b$. Now, by the definition of $\leq$, $eb = bg$ for some $g \in E(S)$, so that

$a = bgf \in H_1^* E(S)$, whence S is (H, I)-factorisable.

Theorem 2.5 Let S be an F-abundant semigroup. Then S is a factorisable semigroup of type-(I, H) if and only if $M_S = H_1^*$, where M_S is the set of the greatest elements in the σ-class of S.

Proof ($\Rightarrow$) If S is a factorisable semigroup of type-(I, H) and F-abundant, from Guo[10] S is also type-(H, I). Then by Theorem 2.3, there exists $h_m \in H_1^*$ such that $m \leq h_m$, for any $m \in M_S$. Notice that $h_m \sigma m$. Thus we have $m = h_m$, hence $M_S \subseteq H_1^*$. On the other hand, since S is an F-abundant semigroup, we have $S = E(S)M_S$. Thus $(E(S), M_S)$ is a factorization of S, and whence by Proposition 2.2, $M_S = H_1^*$.

($\Leftarrow$) Assume that $M_S = H_1^*$. Since S is an F-abundant semigroup, $S = E(S)M_S$, that is, $S = E(S)H_1^*$. Thus S is a factorisable semigroup of type-(I, H).

Recall from [22] that a u-IC quasi-adequate semigroup is defined as an IC quasi-adequate semigroup in which $|U^*(a)| = 1$ for all $a \in S$. Zhang and Chen[22] pointed out that a semigroup S is a u-IC quasi-adequate semigroup if and only if S is isomorphic to a semidirect product of a band by a cancellative monoid; if and only if S is an F-abundant semigroup in which $M_S = H_1^*$. The following theorem gives some characterizations of u-IC quasi-adequate semigroups.

Theorem 2.6 Let S be an IC quasi-adequate monoid. Then the following statements are equivalent:

(1) S is a right univocal factorisable semigroup of type-(I, H).

(2) S is a left univocal factorisable semigroup of type-(H, I).

(3) S is a u-IC quasi-adequate semigroup.

(4) S is isomorphic to a semidirect product of a band by a cancellative monoid.

Proof If S is a left(right) univocal factorisable semigroup of type-(I, H) (-(H, I)), then by Proposition 2.2, S is an F-abundant semigroup, and so by Theorem 2.5, S is a u-IC quasi-adequate semigroup. Now, we need only to prove that (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2). Consider that the proofs of (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are dual, it suffices to show that (3) $\Rightarrow$ (2).

(3) $\Rightarrow$(2) If (3) holds, then by Theorem 2.4, S is (H, I)-factorisable and $|U^*(a)| = 1$. Now let $xe = yf$ with $e, f \in E(S)$ and $x, y \in H_1^*$. Note that $\mathcal{L}^*$ is a right congruence. We observe $xe\mathcal{L}^*1e = e$ and similarly, $yf\mathcal{L}^*f$, hence $x, y \in U^*(xe)$. Thus $x = y$, whence S is left univocal.

§3. Subsemigroups

In this section we describe the full factorisable subsemigroups of an IC quasi-adequate semigroups.

Proposition 3.1 Let S be an IC quasi-adequate monoid with identity 1. If M is a submonoid of H_1^*, then $\downarrow M = \{s \in S \mid s \leq t \text{ for some } t \in M\}$ is a full factorisable subsemigroup of S.

Proof Assume M is a submonoid of H_1^*. Note that the identity 1 of S is the greatest idempotent and $1 \in M$. We have $E(S) \subseteq \downarrow M$ and so $ME(S) \subseteq \downarrow M$. On the other hand, by the definition of $\leq$, $\downarrow M \subseteq ME(S)$. Thus $\downarrow M = ME(S)$ and is a full (H, I)-factorisable subsemigroup of S.

A semigroup S is called a band of cancellative monoid [3] if S is a super abundant semigroup in which $\mathcal{H}^*$ is a congruence.

Lemma 3.2 Let S be an IC quasi-adequate monoid with a band $E(S)$ and ρ be a congruence contained in $\mathcal{H}^*$. Then for every $e \in E(S)$, $e\rho = \{x \in S \mid x\rho e\}$ is a cancellative submonoid of H_e^*. Moreover, $\downarrow 1\rho \subseteq \bigcup_{e\in E(S)} e\rho$.

Proof Certainly, $e\rho$ is a subsemigroup of S. Since $\rho \subseteq \mathcal{H}^*$, it is easy to see that $e\rho \subseteq H_e^*$, so that $e\rho$ is a cancellative submonoid of H_e^*. Let $a \in e\rho$ and $f \in E(S)e$. Then $(fa, fe) \in \rho$, so that $(fa, f) \in \rho \subseteq \mathcal{H}^*$, hence $fa \in f\rho$. What's more, for all $s \in \downarrow 1\rho$ there exist $x \in 1\rho$ such that $s \leq x$, i.e. $s = fx = fx$ with $f \in E(S)$. Thus $s = fx\rho f$,that is, $s \in f\rho \subseteq \bigcup_{e\in E(S)} e\rho$. This can show that $\downarrow 1\rho \subseteq \bigcup_{e\in E(S)} e\rho$.

Proposition 3.3 Let S be an IC quasi-adequate monoid with identity 1. If ρ is a congruence on S which is contained in $\mathcal{H}^*$, then $\downarrow 1\rho$ is a full factorisable subsemigroup of S which is a band of cancellative monoid. In particular, $\downarrow 1\mu$ is a full factorisable subsemigroup of S which is a band of cancellative monoid, where μ is the greatest congruence of S contained in $\mathcal{H}^*$.

Proof Since $\rho \subseteq \mathcal{H}^*$, it is easy to know that $1\rho \subseteq H_1^*$, so that by Proposition 3.1, $\downarrow 1\rho$ is a full factorisable subsemigroup of S. By Proposition 3.2, $\downarrow 1\rho \subseteq \bigcup_{e\in E(S)} e\rho$. Note that $\mathcal{H}^*(S)|_{\downarrow 1\rho} \subseteq \mathcal{H}^*(\downarrow 1\rho)$. We have $a\mathcal{H}^*(\downarrow 1\rho)e$ for all $a \in e\rho \bigcap (\downarrow 1\rho)$. But $E(S) \subseteq \downarrow 1\rho$, now $\downarrow 1\rho$ is a super abundant semigroup.

Now, it remains to verify that $\mathcal{H}^*(\downarrow 1\rho)$ is a congruence on $\downarrow 1\rho$. We firstly prove that $\mathcal{L}^*(\downarrow 1\rho) = \mathcal{L}^*(S) \bigcap (\downarrow 1\rho \times \downarrow 1\rho)$. In fact, if $a, b \in \downarrow 1\rho$ and $a\mathcal{L}^*(\downarrow 1\rho)b$. Clearly, $a\mathcal{L}^*(S)e$ and $b\mathcal{L}^*(S)f$ for some $e, f \in E(S)$, and so $a\mathcal{L}^*(\downarrow 1\rho)e$ and $b\mathcal{L}^*(\downarrow 1\rho)f$ since $\mathcal{L}^*(S)|_{\downarrow 1\rho} \subseteq \mathcal{L}^*(\downarrow 1\rho)$. This shows that $e\mathcal{L}^*(\downarrow 1\rho)f$ and $e\mathcal{L}^*(S)f$. Thus $a\mathcal{L}^*(S)b$. Therefore $\mathcal{L}^*(\downarrow 1\rho) = \mathcal{L}^*(S) \bigcap (\downarrow 1\rho \times \downarrow 1\rho)$. From this and its dual, $\mathcal{H}^*(\downarrow 1\rho) = \mathcal{H}^*(S) \bigcap (\downarrow 1\rho \times \downarrow 1\rho)$. This can show that $\rho\,|_{\downarrow 1\rho} \subseteq \mathcal{H}^*(\downarrow 1\rho)$ since $\downarrow 1\rho \subseteq \bigcup_{e\in E(S)} e\rho$. Let $a, b \in \downarrow 1\rho$ and $(a, b) \in \mathcal{H}^*(\downarrow 1\rho)$, by Proposition 3.1, there exist $a', b' \in 1\rho$ such that $a = a'e$, $b = b'f$ with $e, f \in E(S)$. Moreover, $(a'e, e) \in \rho$ and $(b'f, f) \in \rho$. Therefore, $e = f$ and $(a, b) \in \rho$, that is $\mathcal{H}^*(\downarrow 1\rho) \subseteq \rho\,|_{\downarrow 1\rho}$. Thus $\mathcal{H}^*(\downarrow 1\rho)$ is a congruence on $\downarrow 1\rho$. We complete the proof.

As a consequence of Proposition 3.3, we have:

Corollary 3.4 Any IC quasi-adequate semigroup has a subsemigroup which is a band of cancellative monoids.

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