Chin. Quart. J. of Math.
2013, 28 (3): 383—389

On Skew Power-serieswise nil-Armendariz Rings

ZHANG Wan-ru, PU Zhao-nian

(*College of Mathematics and Statistics, Hexi University, Zhangye* 734000, *China*)

Abstract: For a ring endomorphism α, in this paper we introduce the notion of α-power-serieswise nil-Armendariz rings, which are a generalization of α-power-serieswise Armendariz rings. A number of properties of this generalization are established, and the extensions of α-power-serieswise nil-Armendariz rings are investigated. Which generalizes the corresponding results of nil-Armendariz rings and power-serieswise nil-Armendariz rings.

Key words: nil-Armendariz rings; skew power series ring; nilpotent elements

2000 MR Subject Classification: 16S36, 16W20, 16U99

CLC number: O153.3 **Document code**: A

Article ID: 1002–0462 (2013) 03–0383–07

§1. Introduction

Throughout this paper, all rings are associative rings with identity, unless specified otherwise. For a ring R, we denote by $\mathrm{nil}(R)$ the set of all nilpotent elements of R. $\mathrm{nil}(R)[[x;\alpha]]$ denotes the set of the power series in $R[[x;\alpha]]$ with all coefficients are nilpotent elements.

Recall from [1] that a ring R is called nil-Armendariz if whenever any polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) \in \mathrm{nil}(R)[x]$, then $a_ib_j \in \mathrm{nil}(R)$ for any i and j, where $\mathrm{nil}(R)$ denotes the set of nilpotent elements of R. In [2], Hizem extended the study of nil-Armendariz property to power series rings, and introduced power-serieswise nil-Armendariz rings. A ring R is called power-serieswise nil-Armendariz if whenever power series $f(x) = \sum\limits_{i=0}^{\infty} a_ix^i$, $g(x) = \sum\limits_{j=0}^{\infty} b_jx^j \in R[[x]]$ satisfy $f(x)g(x) \in \mathrm{nil}(R)[[x]]$,

---

Received date: 2012-03-05

Foundation item: Supported by the NNSF of China(10901133); Supported by the Youth Foundation of Hexi University(QN2012-14)

Biography: ZHANG Wan-ru(1980-), male, native of Jinchang, Gansu, a lecturer of Hexi University, engages in rings theory.

then $a_ib_j \in \mathrm{nil}(R)$ for any i and j. Some properties of power-serieswise nil-Armendariz rings and related rings have been studied in [1-4].

For a ring R with a ring endomorphism $\alpha : R \to R$, the skew power series ring $R[[x;\alpha]]$ over R is the ring obtained by giving the power series ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$.

In this paper, we introduce α-power-serieswise nil-Armendariz rings for an endomorphism α of a ring R. We do this by considering the nil-Armendariz property in the skew power series ring $R[[x;\alpha]]$ instead of the ring $R[[x]]$. We will show that α-power-serieswise Armendariz rings are α-power-serieswise nil-Armendariz. Examples are given to show that the converse is not true. Thus α-power-serieswise nil-Armendariz rings are a common generalization of α-power-serieswise Armendariz rings. It is shown that R is α-power-serieswise nil-Armendariz if and only if $\mathrm{nil}(R)$ is an ideal of R and α satisfies the condition $a\alpha(b) \in \mathrm{nil}(R) \Longleftrightarrow ab \in \mathrm{nil}(R)$ for any $a,\ b \in R$. This generalizes the result in [2, Theorem 1] for power-serieswise nil-Armendariz rings. Moreover, we investigate the power-serieswise nil-Armendariz property of the polynomial ring $R[x]$ and power series ring $R[[x]]$. Several known results relating to nil-Armendariz rings and power-serieswise nil-Armendariz rings can be obtained as corollaries of our results.

§2. Skew Power-serieswise nil-Armendariz Rings

Definition 2.1 Let α be an endomorphism of a ring R. R is called an α-power-serieswise nil-Armendariz ring if whenever power series $f(x) = \sum\limits_{i=0}^{\infty} a_ix^i,\ g(x) = \sum\limits_{j=0}^{\infty} b_jx^j \in R[[x;\alpha]]$ satisfy $f(x)g(x) \in \mathrm{nil}(R)[[x;\alpha]]$, then $a_ib_j \in \mathrm{nil}(R)$ for any i and j.

Proposition 2.2 Let R be an α-power-serieswise nil-Armendariz ring. For $a,\ b \in R$, we have the following

(1) If $ab \in \mathrm{nil}(R)$, then $a\alpha^n(b),\ \alpha^n(a)b \in \mathrm{nil}(R)$ for any positive integer n.

(2) If $a\alpha^n(b) \in \mathrm{nil}(R)$ for some positive integer n, then $ab \in \mathrm{nil}(R)$.

(3) If $\alpha^n(a)b \in \mathrm{nil}(R)$ for some positive integer n, then $ab \in \mathrm{nil}(R)$.

Proof (1) Suppose $ab \in \mathrm{nil}(R)$. It is enough to show that $a\alpha(b), \alpha(a)b \in \mathrm{nil}(R)$. Let $f(x) = \alpha(a)x$ and $g(x) = bx$ in $R[[x;\alpha]]$. Then $f(x)g(x) = \alpha(a)\alpha(b)x^2 = \alpha(ab)x^2 \in \mathrm{nil}(R)[[x;\alpha]]$. Since R is α-power-serieswise nil-Armendariz, thus $\alpha(a)b \in \mathrm{nil}(R)$. Note that $ab \in \mathrm{nil}(R)$, then $ba \in \mathrm{nil}(R)$. Hence $\alpha(b)a \in \mathrm{nil}(R)$ by the proof above, and then $a\alpha(b) \in \mathrm{nil}(R)$.

(2) Suppose $a\alpha^n(b) \in \mathrm{nil}(R)$ for some positive integer n. Let $f(x) = ax^n$ and $g(x) = bx$ in $R[[x;\alpha]]$. Then $f(x)g(x) = a\alpha^n(b)x^{n+1} \in \mathrm{nil}(R)[[x;\alpha]]$. Thus $ab \in \mathrm{nil}(R)$ since R is α-power-serieswise nil-Armendariz.

(3) Suppose $\alpha^n(a)b \in \mathrm{nil}(R)$ for some positive integer n. Then $b\alpha^n(a) \in \mathrm{nil}(R)$. Thus $ba \in \mathrm{nil}(R)$ by (2), this implies $ab \in \mathrm{nil}(R)$.

Recall that a ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for any $a,\ b \in R$.

Lemma 2.3 Let R be a ring. Then the following hold

(1) Every α-power-serieswise Armendariz ring is semicommutative.

(2) If R is an α-power-serieswise Armendariz ring, then α is injective and $a\alpha(b) = 0 \Longleftrightarrow ab = 0$ for any $a,\ b \in R$.

(3) Let R be an α-power-serieswise Armendariz ring, $a, b \in R$. If $ab \in \mathrm{nil}(R)$, then $a\alpha^n(b) \in \mathrm{nil}(R)$ for any positive integer n.

(4) Let R be an α-power-serieswise Armendariz ring, $a, b \in R$. If $a\alpha^n(b) \in \mathrm{nil}(R)$ for some positive integer n, then $ab \in \mathrm{nil}(R)$.

Proof (1) and (2) are the results in [5, Theorem 2.4] and [5, Lemma 2.2].

(3) Suppose $ab \in \mathrm{nil}(R)$. It is enough to show that $a\alpha(b) \in \mathrm{nil}(R)$. Let $(ab)^k = 0$, where $k \in \mathbb{N}$. Then $a\alpha(b)\alpha((ab)^{k-1}) = a\alpha(b(ab)^{k-1}) = 0$. Since R is α-power-serieswise Armendariz, thus $a\alpha(b)(ab)^{k-1} = 0$ by (2). Therefore, we have $a\alpha(b)a\alpha(b)\alpha((ab)^{k-2}) = a\alpha(b)a\alpha(b(ab)^{k-2}) = 0$. Continuing this procedure yields that $(a\alpha(b))^k = 0$, this means $a\alpha(b) \in \mathrm{nil}(R)$.

(4) If $a\alpha^n(b) \in \mathrm{nil}(R)$ for some positive integer n, then $\alpha^n(b)a \in \mathrm{nil}(R)$. Thus $\alpha^n(ba) = \alpha^n(b)\alpha^n(a) \in \mathrm{nil}(R)$ by (3). Suppose $[\alpha^n(ba)]^k = \alpha^n((ba)^k) = 0$, then $(ba)^k = 0$ since α is injective. This implies $ba \in \mathrm{nil}(R)$ and then $ab \in \mathrm{nil}(R)$.

Let α be an endomorphism of a ring R. According to [6], α is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for any $a \in R$. R is called an α-rigid ring if there exists a rigid endomorphism α of R.

Let I be an ideal of R. If $\alpha(I) \subseteq I$, then $\overline{\alpha} : R/I \to R/I$ defined by $\overline{\alpha}(a + I) = \alpha(a) + I$ for any $a \in R$ is an endomorphism of the factor ring R/I.

Lemma 2.4 Let R be a ring and α be an endomorphism of R. If α satisfies the condition $a\alpha(b) \in \mathrm{nil}(R) \Longleftrightarrow ab \in \mathrm{nil}(R)$ for any $a, b \in R$, and $\mathrm{nil}(R)$ is an ideal of R, then R is α-power-serieswise nil-Armendariz.

Proof Clearly, $R/\mathrm{nil}(R)$ is reduced. If α satisfies the condition $a\alpha(b) \in \mathrm{nil}(R) \Longleftrightarrow ab \in \mathrm{nil}(R)$, then we have $\bar{a}\overline{\alpha}(\bar{b}) = 0 \Longleftrightarrow \bar{a}\bar{b} = 0$ in $R/\mathrm{nil}(R)$. Thus $R/\mathrm{nil}(R)$ is $\overline{\alpha}$-rigid. Then $R/\mathrm{nil}(R)$ is $\overline{\alpha}$-power-serieswise Armendariz by [5, Proposition 2.3]. Given any $f(x) = \sum\limits_{i=0}^{\infty} a_i x^i, g(x) = \sum\limits_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$ with $f(x)g(x) \in \mathrm{nil}(R)[[x;\alpha]]$, then $\overline{f(x)} \cdot \overline{g(x)} = 0$ in $R/\mathrm{nil}(R)[[x;\overline{\alpha}]]$. Thus $\bar{a_i}\bar{b_j} = 0$ for any i and j since $R/\mathrm{nil}(R)$ is $\overline{\alpha}$-power-serieswise Armendariz. This implies $a_i b_j \in \mathrm{nil}(R)$. Therefore, R is α-power-serieswise nil-Armendariz.

Theorem 2.5 If R is an α-power-serieswise Armendariz ring, then R is α-power-serieswise nil-Armendariz.

Proof Since R is α-power-serieswise Armendariz, thus it is semicommutative by Lemma 2.3. From [7, Lemma 3.1], we have that $\mathrm{nil}(R)$ is an ideal of R. By Lemma 2.3 and Lemma

2.4, the result follows.

Lemma 2.6 Let R be an α-power-serieswise nil-Armendariz ring and $n \geq 2$. If $f_1, f_2, \cdots, f_n \in R[[x;\alpha]]$ with $f_1 f_2 \cdots f_n \in \mathrm{nil}(R)[[x;\alpha]]$, then $a_1 a_2 \cdots a_n \in \mathrm{nil}(R)$, where a_i is any coefficient of f_i for each i.

Proof Suppose $f_1 f_2 \cdots f_n \in \mathrm{nil}(R)[[x;\alpha]]$ and let a_i be any coefficient of f_i. Then $f_1(f_2 \cdots f_n) \in \mathrm{nil}(R)[[x;\alpha]]$ implies $a_1 b \in \mathrm{nil}(R)$ for any coefficient b of $f_2 \cdots f_n$. Hence, $a_1 f_2 \cdots f_n \in \mathrm{nil}(R)[[x;\alpha]]$ and so $(a_1 f_2)(f_3 \cdots f_n) \in \mathrm{nil}(R)[[x;\alpha]]$. Since $a_1 a_2$ is a coefficient of $a_1 f_2$, we have $a_1 a_2 c \in \mathrm{nil}(R)$ for any coefficient c of $f_3 \cdots f_n$. Thus, $a_1 a_2 f_3 \cdots f_n \in \mathrm{nil}(R)[[x;\alpha]]$. Continuing this process, we have $a_1 a_2 \cdots a_n \in \mathrm{nil}(R)$.

Lemma 2.7 Let R be an α-power-serieswise nil-Armendariz ring.

(1) If $a \in \mathrm{nil}(R)$, then $ar, ra \in \mathrm{nil}(R)$ for any $r \in R$.

(2) If $a,\ b,\ c \in \mathrm{nil}(R)$, then $a + bc \in \mathrm{nil}(R)$.

(3) If $a,\ b \in \mathrm{nil}(R)$, then $a - b \in \mathrm{nil}(R)$.

Proof (1) Suppose $a \in \mathrm{nil}(R)$. Then there exists positive integer n such that $a^n = 0$. For any $r \in R$, we have

$$a(1 - rx)[1 + rx + (rx)^2 + \cdots + (rx)^n + \cdots] = a \in \mathrm{nil}(R)[[x;\alpha]].$$

Since R is α-power-serieswise nil-Armendariz, thus $ar \in \mathrm{nil}(R)$ by Lemma 2.6. Then $ra \in \mathrm{nil}(R)$.

(2) Suppose $a,\ b,\ c \in \mathrm{nil}(R)$. By (1), we have bc and $b(a + bc)$ are also nilpotent. Hence $\alpha(b\alpha(a + bc)) \in \mathrm{nil}(R)$ by Proposition 2.2. Note

$$[1 - \alpha(b)x][c + \alpha(a + bc)x] = c + \alpha(a)x - \alpha(b\alpha(a + bc))x^2 \in \mathrm{nil}(R)[[x;\alpha]].$$

Since R is α-power-serieswise nil-Armendariz, we have $1 \cdot \alpha(a + bc) \in \mathrm{nil}(R)$. Therefore $a + bc \in \mathrm{nil}(R)$ by Proposition 2.2.

(3) Suppose $a,\ b \in \mathrm{nil}(R)$. Then $a^2 - ab \in \mathrm{nil}(R)$ by (2). Now by applying (2) several times we can see that $a^2 - ab - ba \in \mathrm{nil}(R)$, hence $a^2 - ab - ba + b^2 \in \mathrm{nil}(R)$. Therefore $(a - b)^2 \in \mathrm{nil}(R)$, which means that $a - b$ is nilpotent.

From Lemma 2.7, we get the following result.

Proposition 2.8 If R is an α-power-serieswise nil-Armendariz ring, then $\mathrm{nil}(R)$ is an ideal of R.

By Proposition 2.2, Lemma 2.4 and Proposition 2.8, we can easily obtain the following result.

Theorem 2.9 Let α be an endomorphism of a ring R. Then R is α-power-serieswise nil-Armendariz if and only if $\mathrm{nil}(R)$ is an ideal of R, and α satisfies the condition $a\alpha(b) \in \mathrm{nil}(R) \Longleftrightarrow ab \in \mathrm{nil}(R)$ for any $a, b \in R$.

Proposition 2.10 Let α be an endomorphism of a ring R and I be an ideal of R with $\alpha(I) \subseteq I$. If $I \subseteq \mathrm{nil}(R)$ and R/I is $\overline{\alpha}$-power-serieswise nil-Armendariz, then R is α-power-serieswise nil-Armendariz.

Proof Let $f(x) = \sum\limits_{i=0}^{\infty} a_i x^i$, $g(x) = \sum\limits_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$ be such that $f(x)g(x) \in \mathrm{nil}(R)[[x;\alpha]]$. Then $(\sum\limits_{i=0}^{\infty} \overline{a_i} x^i)(\sum\limits_{j=0}^{\infty} \overline{b_j} x^j) \in \mathrm{nil}(R/I)[[x;\overline{\alpha}]]$. Since R/I is $\overline{\alpha}$-power-serieswise nil-Armendariz, $(\overline{a_i}\overline{b_j})^{n_{ij}} = 0$ for some positive integer n_{ij}. Thus $a_i b_j \in \mathrm{nil}(R)$. This means that R is α-power-serieswise nil-Armendariz.

Let R_i be a ring and α_i an endomorphism of R_i for each $i \in I$. For the product $\prod\limits_{i\in I} R_i$ of R_i and the endomorphism $\overline{\alpha} : \prod\limits_{i\in I} R_i \mapsto \prod\limits_{i\in I} R_i$ defined by $\overline{\alpha}((a_i)) = (\alpha_i(a_i))$. It can be easily checked that $\prod\limits_{i\in I} R_i$ is $\overline{\alpha}$-power-serieswise nil-Armendariz if and only if each R_i is α_i-power-serieswise nil-Armendariz if the set I is finite.

Let α be an endomorphism of a ring R and $M_n(R)$ be the $n \times n$ full matrix ring over R. Clearly, α can be extended to the endomorphism $\overline{\alpha} : M_n(R) \to M_n(R)$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. Let $T_n(R)$ denotes the $n \times n$ upper triangular matrix ring over R. We also have an extended map $\overline{\alpha}$.

Let R be a ring and denote

$$S_n = \left\{ \left. \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \right| a, a_{ij} \in R \right\},$$

$$R_n = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \right| a_i \in R(1 \le i \le n) \right\}.$$

Then S_n and R_n are rings with addition point-wise and usual matrix multiplication.

Theorem 2.11 Let α be an endomorphism of a ring R. Then the following conditions are equivalent

(1) R is α-power-serieswise nil-Armendariz.

(2) $T_n(R)$ is $\overline{\alpha}$-power-serieswise nil-Armendariz.

(3) S_n is $\overline{\alpha}$-power-serieswise nil-Armendariz.

(4) R_n is $\overline{\alpha}$-power-serieswise nil-Armendariz.

Proof (1) $\Longrightarrow$ (2) Let $I = \{A \in T_n(R) | \text{each diagonal entry of } A \text{ is zero}\}$. Then I is an ideal of $T_n(R)$ with $I \subseteq \mathrm{nil}(T_n(R))$. Clearly, $T_n(R)/I \cong R^n$. So $T_n(R)/I$ is $\overline{\alpha}$-power-serieswise nil-Armendariz. Thus $T_n(R)$ is also $\overline{\alpha}$-power-serieswise nil-Armendariz by Proposition 2.10.

Note that any invariant subring of α-power-serieswise nil-Armendariz rings is α-power-serieswise nil-Armendariz. Thus (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1) is obvious.

§3. Polynomial Ring

Let R be a ring. According to [8], R is nil-semicommutative if whenever $ab \in \mathrm{nil}(R)$, then $arb \in \mathrm{nil}(R)$ for any $r \in R$.

Proposition 3.1 α-power-serieswise nil-Armendariz rings are nil-semicommutative.

Proof Given any $a, b \in R$ with $ab \in \mathrm{nil}(R)$. Then we have $a(1-rx)[1+rx+(rx)^2+\cdots+(rx)^n+\cdots]b = ab \in \mathrm{nil}(R)[[x;\alpha]]$ for any $r \in R$. Since R is α-power-serieswise nil-Armendariz, thus $arb \in \mathrm{nil}(R)$ by Lemma 2.6. This means that R is nil-semicommutative.

Lemma 3.2 Let R be an α-power-serieswise nil-Armendariz ring, then $\mathrm{nil}(R[[x;\alpha]]) \subseteq \mathrm{nil}(R)[[x;\alpha]]$.

Proof Suppose $f(x) \in \mathrm{nil}(R[[x;\alpha]])$, then $f(x)^n = 0$ for some positive integer n. According to Lemma 2.6, we have $a_1a_2\cdots a_n \in \mathrm{nil}(R)$, where a_i is any coefficient of $f(x)$ for any i. In particular, for any coefficient a of $f(x)$, we have $a^n \in \mathrm{nil}(R)$. Thus $a \in \mathrm{nil}(R)$. This means that $f(x) \in \mathrm{nil}(R)[[x;\alpha]]$.

Proposition 3.3 Let R be an α-power-serieswise nil-Armendariz ring such that $\mathrm{nil}(R)[[x;\alpha]] \subseteq \mathrm{nil}(R[[x;\alpha]])$, then $R[[x;\alpha]]$ is nil-semicommutative.

Proof Suppose that $f(x) = \sum\limits_{i=0}^{\infty} a_i x^i, g(x) = \sum\limits_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$ are such that $f(x)g(x) \in \mathrm{nil}(R[[x;\alpha]])$. Then $f(x)g(x) \in \mathrm{nil}(R)[[x;\alpha]]$ by Lemma 3.2. Since R is α-power-serieswise nil-Armendariz, thus $a_ib_j \in \mathrm{nil}(R)$ for any i and j. Given any $h(x) = \sum\limits_{k=0}^{\infty} c_k x^k \in R[[x;\alpha]]$, we have $a_ic_kb_j \in \mathrm{nil}(R)$ since R is nil-semicommutative by Proposition 3.1. Note that $\mathrm{nil}(R)$ is an ideal of R, thus $f(x)h(x)g(x) \in \mathrm{nil}(R)[[x;\alpha]]$. Since $\mathrm{nil}(R)[[x;\alpha]] \subseteq \mathrm{nil}(R[[x;\alpha]])$, thus $f(x)h(x)g(x) \in \mathrm{nil}(R[[x;\alpha]])$. This means that $R[[x;\alpha]]$ is nil-semicommutative.

Recall that if α is an endomorphism of a ring R, then the map α can be extended to an endomorphism of the polynomial ring $R[x]$ defined by $\sum\limits_{i=0}^{n} a_i x^i \mapsto \sum\limits_{i=0}^{n} \alpha(a_i) x^i$. We denote the extended map $R[x] \mapsto R[x]$ by $\overline{\alpha}$.

Theorem 3.4 Let R be an α-power-serieswise nil-Armendariz ring with $\mathrm{nil}(R)[x] = \mathrm{nil}(R[x])$. If $\alpha^t = I_R$ for some positive integer t, then $R[x]$ is $\overline{\alpha}$-power-serieswise nil-Armendariz.

Proof Suppose that $p(y) = \sum\limits_{i=0}^{\infty} f_i y^i, q(y) = \sum\limits_{j=0}^{\infty} g_j y^j \in R[x][[y;\overline{\alpha}]]$ are such that $p(y)q(y) \in$

$\mathrm{nil}(R[x])[[y;\overline{\alpha}]]$. Let $f_i=a_{i0}+a_{i1}x+\cdots+a_{ik_i}x^{k_i}, g_j=b_{j0}+b_{j1}x+\cdots+b_{jl_j}x^{l_j}$ for any i and j, where $a_{i0},\ a_{i1},\ \cdots,\ a_{ik_i},\ b_{j0},\ b_{j1},\ \cdots,\ b_{jl_j}\in R$. We claim that $f_ig_j\in \mathrm{nil}(R[x])$ for all i and j. Let $k_n=\sum\limits_{i=0}^{n}\deg f_i+\sum\limits_{j=0}^{n}\deg g_j+1$, where the degree is as polynomials in $R[x]$ and the degree of zero polynomial is taken to be 0. Since $p(y)q(y)\in \mathrm{nil}(R[x])[[y;\overline{\alpha}]]$ and $\mathrm{nil}(R)[x]=\mathrm{nil}(R[x])$, we have

$$f_0g_0\in \mathrm{nil}(R)[x];$$

$$f_0g_1+f_1\overline{\alpha}(g_0)\in \mathrm{nil}(R)[x];$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_0g_n+f_1\overline{\alpha}(g_{n-1})+\cdots+f_{n-1}\overline{\alpha}^{n-1}(g_1)+f_n\overline{\alpha}^n(g_0)\in \mathrm{nil}(R)[x];$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Let $f(x)=f_0(x^t)+f_1(x^t)x^{tk_1+1}+\cdots+f_m(x^t)x^{mtk_m+m}+\cdots$ and $g(x)=g_0(x^t)+g_1(x^t)x^{tk_1+1}+\cdots+g_n(x^t)x^{ntk_n+n}+\cdots$. Then the set of coefficients of the $f_i(x)$(respectively, $g_j(x)$) equals the set of coefficients of $f(x)$(respectively, $g(x)$). Since $\alpha^t=I_R$, we have

$$f(x)g(x)=f_0(x^t)g_0(x^t)+[f_0(x^t)g_1(x^t)+f_1(x^t)\overline{\alpha}(g_0(x^t))]x^{tk_1+1}+\cdots\in \mathrm{nil}(R)[[x;\alpha]].$$

Since R is α-power-serieswise nil-Armendariz, thus $a_{il}b_{jr}\in \mathrm{nil}(R)$ for any i and j. Note that $\mathrm{nil}(R)$ is an ideal of R, thus $f_ig_j\in \mathrm{nil}(R)[x]$. Since $\mathrm{nil}(R)[x]=\mathrm{nil}(R[x])$, thus $f_ig_j\in \mathrm{nil}(R[x])$. This implies that $R[x]$ is $\overline{\alpha}$-power-serieswise nil-Armendariz.

[References]

[1] ANTOINE R. Nilpotent elements and Armendariz rings[J]. J Algebra, 2008, 319: 3128–3140.

[2] HIZEM S. A note on nil power serieswise Armendariz rings[J]. Rend Circ Mat Palermo, 2010, 59: 87–99.

[3] FAN Wei-li, WANG Hui. Special properties of formal triangular matrix rings[J]. Chin Quart J of Math, 2008, 23(4): 625-632.

[4] ZHANG Fu-li. A note on the supersemiprime rings and supersemiprime radical[J]. Chin Quart J of Math, 2002, 17(3): 105-106.

[5] NASR-ISFAHANI A R, MOUSSAVI A. On skew power serieswise Armendariz rings[J]. Comm Algebra, 2011, 39(9): 3114–3132.

[6] HONG C Y, KIM N K, KWAK T K. Ore extensions of Baer and p.p.-rings[J]. J Pure Appl Algebra, 2000, 151(3): 215–226.

[7] LIU Zhong-kui, ZHAO Ren-yu. On weak Armendariz rings[J]. Comm Algebra, 2006, 34(7): 2607–2616.

[8] CHEN Wei-xing. On nil-semicommutative rings[J]. Thai J Math, 2011, 9(1): 39–47.