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Majorization and Fekete-Szegö Problems for Multivalent Meromorphic Functions Associated with the Mittag-Leffler Function

LONG Pin-hong[1], *GANGADHARAN Murugusundaramoorthy*[2], *HAN Hui-li*[3], *WANG Wen-shuai*[1]

(1. School of Mathematics and Statistics, Ningxia University, Yinchuan 750000, China; 2. School of Advanced Sciences, Vellore Institute of Technology, Deemed to be University, Vellore 632014, India; 3. School of Mathematics and Computer Science, Ningxia Normal University, Guyuan 756099, China)

Abstract: The main object of this present paper is to investigate the problem of majorization of certain class of multivalent meromorphic functions of complex order involving Mittag-Leffler function. Meanwhile, for this subclass the corresponding coefficient estimates and some Fekete-Szegö type inequalities are obtained. Moreover we point out some new or known consequences of our main results.

Keywords: Meromorphic function; Majorization problem; Hadamard product (convolution); Mittag-Leffler function; Fekete-Szegö inequality

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§1. Introduction

Denote by $\mathcal{A}$ the class of functions whose elements are of the following form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$ and normalized by the conditions $f(0)=0=f'(0)-1$.

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Biographies: LONG Pin-hong (1978-), male, native of Chenzhou, Hunan, associate professor of Ningxia University, engages in complex analysis and its application; GANGADHARAN Murugusundaramoorthy (1967-), male, native of Vellore, professor of Vellore Institute of Technology, engages in complex analysis; HAN Hui-li (1972-), female, native of Yinchuan, Ningxia, professor of Ningxia Normal University, engages in complex analysis and its application; WANG Wen-shuai (1980-), male, native of Guyuan, Ningxia, associate professor of Ningxia University, engages in complex analysis and its application. Corresponding author LONG Pin-hong: longph@nxu.edu.cn.

Let f and g be analytic functions in the unit disc $\Delta=\{z\in\mathbb{C}:|z|<1\}$. Due to MacGregor [16] (also see [20]), we say that f is majorized by g in Δ and write

$$f(z)\ll g(z), \quad (z\in\Delta), \tag{1.2}$$

if there exists an analytic function ϕ in Δ such that

$$|\phi(z)|<1 \text{ and } f(z)=\phi(z)g(z), \quad (z\in\Delta). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions. Also, we say that f is subordinate to g, which is denoted by $f\prec g$ (see [17]), if there exists a Schwarz function ω which is analytic in Δ with $\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in\Delta$, such that

$$f(z)=g(\omega(z)), \quad (z\in\Delta).$$

Furthermore, if the function g is univalent in Δ, we have

$$f\prec g \Longleftrightarrow f(0)=g(0) \text{ and } f(\Delta)\subset g(\Delta).$$

Denote by $\mathcal{S}^*(\tau)$ and $\mathcal{C}(\tau)$ the class of starlike and convex functions of complex order $\tau(\tau\in\mathbb{C}\setminus\{0\}=\mathbb{C}^*)$, satisfying the following conditions

$$\frac{f(z)}{z}\neq 0 \text{ and } \mathfrak{Re}\left[1+\frac{1}{\tau}\left(\frac{zf'(z)}{f(z)}-1\right)\right]>0$$

and

$$f'(z)\neq 0 \text{ and } \mathfrak{Re}\left[1+\frac{1}{\tau}\left(\frac{zf''(z)}{f'(z)}\right)\right]>0, \quad (z\in\Delta),$$

respectively. Further,

$$\mathcal{S}^*((1-\alpha)cos\lambda\ e^{-i\lambda})=\mathcal{S}^*(\alpha,\lambda), \quad (|\lambda|<\frac{\pi}{2},\ 0\le\alpha\le 1)$$

and

$$\mathcal{S}^*(cos\lambda\ e^{-i\lambda})=\mathcal{S}^*(\lambda), \quad (|\lambda|<\frac{\pi}{2}),$$

where one denotes by $\mathcal{S}^*(\alpha,\lambda)$ the class of $\lambda-$Spiral-like function of order α investigated by Libera [14] and by $\mathcal{S}^*(\lambda)$ Spiral-like functions introduced by Špaček [26] (see [23]).

Let $\mathcal{P}$ be the class of all analytic functions $\ell(z)$ of the following form

$$\ell(z)=1+\sum_{n=1}^{\infty}c_n z^n, \quad (z\in\Delta)$$

satisfying $\Re\ell(z)>0$ and $\ell(0)=1$.

Lemma 1.1. *(See [5, 11]) Assume that the function $\ell(z)\in\mathcal{P}$, then the sharp estimates $|c_n|\leq 2$ $(n\in\mathbb{N})$ are true. In Particular, the equality holds for all n for the next function*

$$\ell(z)=\frac{1+z}{1-z}=1+\sum_{n=1}^{\infty}2z^n.$$

Lemma 1.2. *([15]) Assume that the function $\ell(z)\in\mathcal{P}$, then*

$$|c_2-\hbar c_1^2|\leq 2\max\{1,|2\hbar-1|\},\quad (\hbar\in\mathbb{C}).$$

Specially, the sharp result holds for the next functions

$$\ell(z)=\frac{1+z}{1-z}\ \textit{and}\ \ell(z)=\frac{1+z^2}{1-z^2},\quad (z\in\Delta).$$

Lemma 1.3. *([15]) Assume that the function $\ell(z)\in\mathcal{P}$ and $\kappa\in\mathbb{R}$, then*

$$|c_2-\kappa c_1^2|\leq\begin{cases}-4\kappa+2, & \textit{if}\ \ \kappa\leq 0,\\ 2, & \textit{if}\ 0\leq\kappa\leq 1,\\ 4\kappa-2, & \textit{if}\ \ \kappa\geq 1.\end{cases}$$

For $\kappa<0$ or $\kappa>1$, the inequality holds literally if and only if $\ell(z)=\frac{1+z}{1-z}$ or one of its rotations. If $0<\kappa<1$, the inequality holds literally if and only if $\ell(z)=\frac{1+z^2}{1-z^2}$ or one of its rotations. In particular, if $\kappa=0$, then the sharp result holds for the next function

$$\ell(z)=\left(\frac{1}{2}+\frac{\xi}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\xi}{2}\right)\frac{1-z}{1+z},\quad (0\leq\xi\leq 1),$$

or one of its rotations. If $\kappa=1$, then the sharp result holds for the next function

$$\frac{1}{\ell(z)}=\left(\frac{1}{2}+\frac{\xi}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\xi}{2}\right)\frac{1-z}{1+z},\quad (0\leq\xi\leq 1),$$

or one of its rotations. If $0<\kappa<1$, then the upper bound is sharp as the followings

$$|c_2-\kappa c_1^2|+\kappa|c_1|^2\leq 2,\quad (0<\kappa\leq\frac{1}{2})$$

and

$$|c_2-\kappa c_1^2|+(1-\kappa)|c_1|^2\leq 2,\quad (\frac{1}{2}<\kappa<1).$$

Let Σ_p be the class of $p-$valently meromorphic functions which are analytic and univalent in the punctured unit disk

$$\Delta^*=\{z\in\mathbb{C}:0<|z|<1\}=\Delta\setminus\{0\}$$

of the form

$$f(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}a_nz^{n-p} \tag{1.4}$$

with a simple pole at the origin.

For functions $f_j(z)=\frac{1}{z^p}+\sum\limits_{n=1}^{\infty}a_{n,j}z^{n-p}$ $(j=1,2,\ p\in\mathbb{N})$ in Σ_p, the convolution or Hadamard product of two functions $f_1,f_2\in\Sigma_p$ is denoted by f_1*f_2 and defined as

$$(f_1*f_2)(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}a_{n,1}a_{n,2}z^{n-p}=(f_2*f_1)(z). \tag{1.5}$$

The function $\mathbf{E}_\alpha(z)$ was introduced by Mittag-Leffler [18,19] and is, therefore, known as the Mittag-Leffler function. A more general function $\mathbf{E}_{\alpha,\beta}$ with two parameters α,β, which generalizes $E_\alpha(z)$, was introduced by Wiman [28,29] and denoted by

$$\mathbf{E}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^k}{\Gamma(\alpha n+\beta)},\quad (z,\alpha,\beta\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0). \tag{1.6}$$

The Mittag-Leffler function arises naturally in the solutions of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found, e.g. in [3,4,8–10,13]. Observe that Mittag-Leffler function $\mathbf{E}_{\alpha,\beta}(z)$ does not belong to the class $\mathcal{A}$. Therefore, it is natural to consider the following normalization of Mittag-Leffler functions as below :

$$E_{\alpha,\beta}(z)=z\Gamma(\beta)\mathbf{E}_{\alpha,\beta}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}z^n, \tag{1.7}$$

which holds for complex parameters α, β and $z\in\mathbb{C}$. There has been a growing focus on Mittag-Leffler-type functions in recent years based on the growth of possibilities for their application for probability, applied problems, statistical and distribution theory, among others. In most of our work related to Mittag-Leffler functions, we study the geometric properties, such as the convexity, close-to-convexity and starlikeness. Recent studies on the Mittag-Leffler function $E_{\alpha,\beta}(z)$ can be seen in [21]. In fact, the function given by (1.6) is not within the class Σ. Based on the above reason, this special function is then normalized as follows:

$$\Omega_\beta^\alpha(z)=z^{-1}\Gamma(\beta)E_{\alpha,\beta}(z)=z^{-1}+\sum_{n=0}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)}z^n.$$

Srivastava and Tomovski [25] introduced the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ of the form:

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\delta}}{\Gamma(\alpha n+\beta)\ n!}z^n, \tag{1.8}$$

where $\beta,\gamma\in\mathbb{C}$, $\Re(\alpha)>\max\{0;\Re(\delta)-1\}$, $\Re(\delta)>0$, $\Re(\alpha)=0$ when $\Re(\delta)=1$ with $\beta\neq0$ and $(\upsilon)_m$ denotes the Pochhammer symbol (or the shifted factorial) as a product of m factors by

$$(\upsilon)_m=\begin{cases}1, & m=0,\\ \upsilon(\upsilon+1)(\upsilon+2)\cdots(\upsilon+m-1), & m\in\mathbb{N}.\end{cases}$$

Then, Srivastava and Tomovski [25] proved that the function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ defined by (1.8) is an entire function in the complex z-plane. Further, we define the function $E_{p,\alpha,\beta}^{\gamma,\delta}(z)$ by

$$E_{p,\alpha,\beta}^{\gamma,\delta}(z)=z^{-p}\Gamma(\beta)E_{\alpha,\beta}^{\gamma,\delta}(z).$$

Corresponding to the function $E_{p,\alpha,\beta}^{\gamma,\delta}(z)$ defined by (1.5), we introduce a linear operator

$$\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}:\Sigma_p\to\Sigma_p$$

by

$$\begin{aligned}\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}f(z)=E_{p,\alpha,\beta}^{\gamma,\delta}(z)*f(z)&=z^{-p}+\sum_{n=1}^{\infty}\frac{\Gamma(\gamma+n\delta)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\alpha n+\beta)\ n!}a_nz^{n-p}\\&=z^{-p}+\sum_{n=1}^{\infty}a_nM_n(\alpha,\beta,\gamma,\delta)z^{n-p},\end{aligned}$$

where $\Re(\alpha)>\max\{0;\Re(\delta)-1\}$, $\Re(\delta)>0$ and $\Re(\alpha)=0$ when $\Re(\delta)=1$ with $\beta\neq 0$, and

$$M_n(\alpha,\beta,\gamma,\delta)=\frac{\Gamma(\gamma+n\delta)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\alpha n+\beta)\ n!}:=M_n.$$

Here, we also point out that

$$M_1=\frac{\Gamma(\gamma+\delta)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\alpha+\beta)},\quad M_2=\frac{\Gamma(\gamma+2\delta)\Gamma(\beta)}{2\Gamma(\gamma)\Gamma(2\alpha+\beta)}\text{ and }M_3=\frac{\Gamma(\gamma+3\delta)\Gamma(\beta)}{6\Gamma(\gamma)\Gamma(3\alpha+\beta)}.\tag{1.9}$$

Also, it is easily verified from (1.6) that

$$z\left(\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}f(z)\right)'=\frac{\gamma}{\delta}\mathfrak{M}_{p,\alpha,\beta}^{\gamma+1,\delta}f(z)-(\frac{\gamma+p\delta}{\delta})\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}f(z).\tag{1.10}$$

Stimulated by the earlier works on majorization problem for the class of analytic starlike functions that have been investigated by MacGregor [16], and Altintas et al. [1], lately Goyal and Goswami [12] extended their results into the class of meromorphic functions by making use of certain integral operator. The Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ for $f\in\mathcal{S}$ is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [7] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. Ever since, the functional has received great attention, particularly in the study of many subclasses of the family of univalent functions (see [24]). This topic has become of considerable interest among researchers in Geometric Function Theory, for the meromorphic functions in recent past (see, for example, [2, 6, 22, 27]). In the present paper we introduce a new class of $p-$valently meromorphic starlike functions of complex order associated with the generalized Mittag-Leffler functions and investigate a majorization problem for functions that belong to the class $\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$. Further, we discuss Fekete-Szegö functional for $f\in\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$.

Definition 1.1. *A function $f(z)\in\Sigma_p$ is said to in the class $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$ of meromorphic functions of complex order $\tau\in\mathbb{C}^*$ in Δ^* if and only if*

$$1-\frac{1}{\tau}\left(\frac{z\left(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)\right)'}{\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)}+p\right)\prec\frac{1+Az}{1+Bz}, \tag{1.11}$$

where $z\in\Delta^*$, $-1\le B<A\le 1$.

Example 1.1. *If we put*

$$\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,1,-1)=\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau),$$

then $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau)$ denotes the class of functions $f\in\Sigma_p$ satisfying the following inequality:

$$\Re\left[1-\frac{1}{\tau}\left(\frac{z\left(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)\right)'}{\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)}+p\right)\right]>0,$$

where $z\in\Delta^*$ and $\tau\in\mathbb{C}^*$.

Example 1.2. *When $\tau=(p-\vartheta)cos\lambda\ e^{-i\lambda}$ $(|\lambda|<\frac{\pi}{2},\ 0\le\vartheta<p)$, the class*

$$\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau)=\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}((p-\vartheta)cos\lambda\ e^{-i\lambda})\equiv\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\vartheta,\lambda)$$

is called the generalized class of λ-spiral-like functions $f(z)$ of order ϑ $(0\le\vartheta<p)$ satisfying

$$\Re\left(e^{i\lambda}\frac{z\left(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)\right)'}{\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)}\right)<-\vartheta\ cos\lambda.$$

Example 1.3. *Once we set $\tau=p-\vartheta$ $(0\le\vartheta<p)$, the class*

$$\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau)=\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(p-\vartheta)\equiv\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\vartheta)$$

is named the generalized class of $p-$valently meromorphic starlike functions $f(z)$ of order ϑ $(0\le\vartheta<p)$ satisifying

$$\Re\left(\frac{z\left(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)\right)'}{\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)}\right)<-\vartheta.$$

§2. A majorization problem for the class $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$

In this section we shall restrict our attention to the case of $\alpha,\beta,\gamma,\delta\in\mathbb{R}_+$. Then, we deal with the majorization problem for the class $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$ as follow.

Theorem 2.1. *Let the function $f(z)\in\Sigma_p$ and $g(z)\in\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$. If $\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)$ is majorized by $\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)$ in Δ^*, then*

$$|(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z))'|\le|(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z))'|,\quad(|z|\le r_1), \tag{2.1}$$

where r_1 is the smallest positive root of the equation

$$|\gamma B-\delta\tau(A-B)|\ r^3-(2\delta|B|+\gamma)r^2-\{2\delta+|\gamma B-\delta\tau(A-B)|\}r+\gamma=0, \tag{2.2}$$

and $p\in\mathbb{N}_0$, $\tau\in\mathbb{C}^*$, $-1\le B<A\le 1$.

Proof. Since $g(z)\in\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$, we readily obtain from (1.11) that

$$1-\frac{1}{\tau}\left[\frac{z\left(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)\right)'}{\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)}+p\right]=\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where w belongs to the well-known class of bounded analytic functions in Δ and

$$w(0)=0 \text{ and } |w(z)|\le|z|, \quad (z\in\Delta). \tag{2.4}$$

From (2.3), we get

$$\frac{z\left(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)\right)'}{\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)}=-\frac{p+[pB+\tau(A-B)]w(z)}{1+Bw(z)}. \tag{2.5}$$

Using (1.10) in (2.5), we derive

$$\left|\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)\right|=\left|\frac{\gamma[1+Bw(z)]}{\gamma+\{\gamma B-\delta\tau(A-B)\}w(z)}\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}g(z)\right|. \tag{2.6}$$

Hence, by making use of (2.4), we deduce

$$|\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)|\le\frac{\gamma[1+|B|\ |z|]}{\gamma-|\gamma B-\delta\tau(A-B)|\ |z|}|(\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}g(z)|. \tag{2.7}$$

Since $\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)$ is majorized by $\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)$ in Δ^* from (1.3), we have

$$\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)=\phi(z)\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z).$$

Differentiating the above equation w.r.t z and multiplying by z, we infer

$$z(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z))'=z\phi'(z)\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)+z\phi(z)(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z))'.$$

By applying (1.10), we get

$$\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}f(z)=z\phi'(z)\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)+\frac{\gamma}{\delta}\phi(z)(\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}g(z)). \tag{2.8}$$

Noting that the Schwarz function $\phi(z)$ satisfies (see Nehari [20])

$$|\phi'(z)|\le\frac{1-|\phi(z)|^2}{1-|z|^2} \tag{2.9}$$

and using (2.7) and (2.9) in (2.8), we obtain

$$|\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}f(z)|\le|\phi(z)|+\frac{(1-|\phi(z)|^2)|z|}{(1-|z|^2)}\cdot\frac{\delta[1+|B|\ |z|]}{\gamma-|\gamma B-\delta\tau(A-B)|\ |z|}|\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}g(z)|,$$

which upon setting

$$|z|=r \text{ and } |\phi(z)|=\rho, \quad (0\le\rho\le1)$$

leads us to the next inequality

$$|\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}f(z)|\le\Lambda_1(r,\rho)|\mathfrak{M}^{\gamma+1,\delta}_{p,\alpha,\beta}g(z)|, \tag{2.10}$$

where the function $\Lambda_1(r,\rho)$ is given by

$$\Lambda_1(r,\rho)=\rho+\frac{r(1-\rho^2)\delta[1+|B|r]}{(1-r^2)\{\gamma-|\gamma B-\delta\tau(A-B)|r\}}.$$

In order to determine the bound of $\Lambda_1(r,\rho)$, we have to choose

$$\begin{aligned}r_1&=\max\{r\in(0,1):\Lambda_1(r,\rho)\le1,\ \rho\in[0,1]\}\\&=\max\{r\in(0,1):\Lambda_2(r,\rho)\ge0,\ \rho\in[0,1]\},\end{aligned}$$

where

$$\Lambda_2(r,\rho)=(1-r^2)\{\gamma-|\gamma B-\delta\tau(A-B)|r\}-r(1+\rho)\delta[1+|B|r].$$

Obviously, for $\rho=1$, the function $\Lambda_2(r,\rho)$ takes its minimum value, namely

$$\min\{\Lambda_2(r,\rho):\rho\in[0,1]\}=\Lambda_2(r,1)=\Lambda_2(r),$$

where

$$\Lambda_2(r)=(1-r^2)\{\gamma-|\gamma B-\delta\tau(A-B)|r\}-2r\delta[1+|B|r].$$

Furthermore, if $\Lambda_2(0)=\gamma>0$ and $\Lambda_2(1)=-2\delta[1+|B|]<0$, then there exists a positive constant r_1 such that $\Lambda_2(r)\ge0$ for all $r\in(0,r_1]$, where r_1 the smallest positive root of the equation. Then, it completes the proof of Theorem 2.1. □

By taking $A=1$ and $B=-1$ in Theorem 2.1 we state the following corollary without proof.

Corollary 2.1. *Let the function $f(z)\in\Sigma_p$ and $g(z)\in\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau)$ if $\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z)$ is majorized by $\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z)$ in Δ^*, then*

$$|(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}f(z))'|\le|(\mathfrak{M}^{\gamma,\delta}_{p,\alpha,\beta}g(z))'|, \quad (|z|\le r_1),$$

where r_1 is the smallest positive root of the equation

$$|\gamma+2\delta\tau|\ r^3-(2\delta+\gamma)r^2-\{2\delta+|\gamma+2\delta\tau|\}r+\gamma=0,$$

and $p\in\mathbb{N}_0,\ \tau\in\mathbb{C}^*$.

Remark 2.1. *Fixing the parameters $\tau=(p-\vartheta)cos\ \lambda e^{-i\lambda}$ $(|\lambda|<\frac{\pi}{2},\ \vartheta\ (0\le\vartheta<p)$, and further letting $\lambda=0$ in Corollary 2.1, one can state the majorization results for the function classes $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\vartheta,\lambda)$ and $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\vartheta)$ defined in Examples 1.2 and 1.3, respectively.*

§3. Fekete-Szegö inequalities for the function class $\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$

From now on, we study the coefficient estimates and Fekete-Szegö inequalities for the class $\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$. At first we establish the following theorem for coefficient bounds.

Theorem 3.1. *If $f(z)\in\Sigma_p$ belongs to the class $\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$, then*

$$|a_1|\leq\frac{|(A-B)\tau|}{|M_1|}, \tag{3.1}$$

$$|a_2|\leq\frac{\Xi(\tau,A,B)|(A-B)\tau|}{2|M_2|} \tag{3.2}$$

and

$$|a_3|\leq\frac{[3\Xi(\tau,A,B)|(A-B)\tau|+2(4+2B+B^2)+2|(A-B)\tau|^2]|(A-B)\tau|}{6|M_3|},$$

where

$$\Xi(\tau,A,B)=B+2+|(A-B)\tau| \tag{3.3}$$

and M_1,M_2,M_3 are given by (1.9).

Proof. Assume that $f(z)\in\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$. Then, by Definition 1.1 there exists an analytic function $\omega(z):\Delta\to\Delta$ with $\omega(0)=0$ such that

$$1-\frac{1}{\tau}\left[\frac{z\left(\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}f(z)\right)'}{\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}f(z)}+p\right]=\frac{1+Aw(z)}{1+Bw(z)}. \tag{3.4}$$

Expanding the left side of (3.4), we get that

$$1-\frac{1}{\tau}\left[\frac{z\left(\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}f(z)\right)'}{\mathfrak{M}_{p,\alpha,\beta}^{\gamma,\delta}f(z)}+p\right]=1+\frac{M_1a_1}{\tau}z+\frac{2M_2a_2-M_1^2a_1^2}{\tau}z^2$$
$$+\frac{3M_3a_3-3M_2a_2M_1a_1+M_1^3a_1^3}{\tau}z^3+\cdots. \tag{3.5}$$

Besides, if we denote the function $\ell\in\mathcal{P}$ by

$$\ell(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+\sum_{n=1}^{\infty}c_nz^n,\quad (z\in\Delta), \tag{3.6}$$

then from (30) we know that

$$\omega(z)=\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)z^3+\ldots,\quad (z\in\Delta), \tag{3.7}$$

such that

$$\frac{1+A\omega(z)}{1+B\omega(z)}=1+\frac{(A-B)c_1z}{2}+\frac{A-B}{2}\left(c_2-\frac{B+1}{2}c_1^2\right)z^2$$

$$+\frac{A-B}{2}\left(c_3-\frac{B+2}{2}c_1c_2+\frac{1+B+B^2}{4}c_1^3\right)z^3+\ldots,\quad (z\in\Delta). \tag{3.8}$$

Therefore, together (3.4-3.5) with (3.8), we remark that

$$\frac{M_1a_1}{\tau}=\frac{(A-B)c_1}{2}, \tag{3.9}$$

$$\frac{2M_2a_2-M_1^2a_1^2}{\tau}=\frac{A-B}{2}\left(c_2-\frac{B+1}{2}c_1^2\right) \tag{3.10}$$

and

$$\frac{3M_3a_3-3M_2a_2M_1a_1+M_1^3a_1^3}{\tau}=\frac{A-B}{2}\left(c_3-\frac{B+2}{2}c_1c_2+\frac{1+B+B^2}{4}c_1^3\right). \tag{3.11}$$

From (3.9) and (3.10), we derive that

$$a_1=\frac{(A-B)\tau c_1}{2M_1} \tag{3.12}$$

and

$$a_2=\frac{(A-B)^2\tau^2c_1^2}{8M_2}+\frac{(A-B)\tau}{4M_2}\left(c_2-\frac{B+1}{2}c_1^2\right). \tag{3.13}$$

Thereby, from (3.12-3.13) and Lemma 1.1, we obtain that

$$|a_1|\le\frac{|(A-B)\tau|}{|M_1|} \tag{3.14}$$

and

$$|a_2|\le\frac{(B+2+|(A-B)\tau|)|(A-B)\tau|}{2|M_2|}. \tag{3.15}$$

Moreover, by (3.11-3.15) and Lemma 1.1, we get that

$$|a_3|\le\frac{[3(B+2+|(A-B)\tau|)|(A-B)\tau|+2(4+2B+B^2)+2|(A-B)\tau|^2]|(A-B)\tau|}{6|M_3|}.$$

This theorem is proved.□

Next, we consider Fekete-Szegö problems for the class $\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$.

Theorem 3.2. *Let $f(z)\in\sum_p$ belong to the class $\mathcal{M}_{p,\alpha,\beta}^{\gamma,\delta}(\tau,A,B)$ and $\varrho\in\mathbb{C}$, then*

$$|a_2-\varrho a_1^2|\le\frac{|(A-B)\tau|\max\{1,|2\hbar-1|\}}{2|M_2|},$$

where

$$\hbar=\frac{B+1}{2}+\frac{\varrho(A-B)\tau M_2}{M_1^2}-\frac{(A-B)\tau}{2}.$$

Proof. From (3.12) and (3.13) we can deduce that

$$a_2-\varrho a_1^2=\frac{(A-B)\tau}{4M_2}\left[c_2-\left(\frac{B+1}{2}+\frac{\varrho(A-B)\tau M_2}{M_1^2}-\frac{(A-B)\tau}{2}\right)c_1^2\right].$$

Hence, from Lemma 1.2, we imply that

$$|\,a_2-\varrho a_1^2\,|\leq\frac{|(A-B)\tau|\max\{1,|2\hbar-1|\}}{2|M_2|},$$

where

$$\hbar=\frac{B+1}{2}+\frac{\varrho(A-B)\tau M_2}{M_1^2}-\frac{(A-B)\tau}{2}.$$

Consequently, the proof is completed.□

If we choose $\varrho=1$ and $\varrho=0$ in Theorem 3.2, respectively, then we have the following corollary for Fekete-Szegö problem and the bound of coefficient a_2.

Corollary 3.1. *Let $f(z)\in\sum_p$ belong to the class $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$, then*

$$|\,a_2-a_1^2\,|\leq\frac{|(A-B)\tau|\max\{1,|2\hbar-1|\}}{2|M_2|},$$

where

$$\hbar=\frac{B+1}{2}+\frac{(A-B)\tau M_2}{M_1^2}-\frac{(A-B)\tau}{2}$$

and

$$|\,a_2\,|\leq\frac{|(A-B)\tau|\max\{1,|B-(A-B)\tau|\}}{2|M_2|}.$$

Furthermore, if we take all the parameters $\alpha,\beta,\gamma,\delta\in\mathbb{R}_+$ so that $M_n>0$ for all $n\in\mathbb{N}$, then together $\tau\in\mathbb{R}_+$ with Lemma 1.3 and (3.9) we provide Fekete-Szegö type inequalities for the class $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$.

Theorem 3.3. *Let $f(z)\in\sum_p$ belong to the class $\mathcal{M}^{\gamma,\delta}_{p,\alpha,\beta}(\tau,A,B)$ and $\varrho\in\mathbb{R}$, then*

$$|\,a_2-\varrho a_1^2\,|\leq\begin{cases}\frac{(A-B)\tau}{M_2}\times\left\{(A-B)\tau[1-\frac{2\varrho M_2}{M_1^2}]-B\right\}, & \text{if } \varrho\leq\Upsilon_1,\\ \frac{(A-B)\tau}{M_2}, & \text{if } \Upsilon_1\leq\varrho\leq\Upsilon_2,\\ \frac{(A-B)\tau}{M_2}\times\left\{(A-B)\tau[\frac{2\varrho M_2}{M_1^2}-1]+B\right\}, & \text{if } \varrho\geq\Upsilon_2,\end{cases}$$

where

$$\Upsilon_1=\frac{[(A-B)\tau-B-1]M_1^2}{2(A-B)\tau M_2}$$

and

$$\Upsilon_2=\frac{[(A-B)\tau-B+1]M_1^2}{2(A-B)\tau M_2}.$$

Moreover, if $\Upsilon_1<\varrho<\Upsilon_2$*, then there exists the sharp upper bounds as the followings*

$$|a_2-\varrho a_1^2|+\left\{(A-B)\tau(2\varrho M_2-M_1^2)+(B+1)M_1^2\right\}\times\frac{|a_1|^2}{2(A-B)\tau M_2}\le\frac{(A-B)\tau}{2M_2}$$

for $\Upsilon_1<\varrho\le\widetilde{\Upsilon}$ *and*

$$|a_2-\varrho a_1^2|+\left\{(A-B)\tau(M_1^2-2\varrho M_2)-(B-1)M_1^2\right\}\times\frac{|a_1|^2}{2(A-B)\tau M_2}\le\frac{(A-B)\tau}{2M_2}$$

for $\widetilde{\Upsilon}<\varrho<\Upsilon_2$*, where*

$$\widetilde{\Upsilon}=\frac{[(A-B)\tau-B]M_1^2}{2(A-B)\tau M_2}.$$

§4. Concluding remarks

Up to this step we have introduced a new meromorphic function subclass which is related to the Mittag-Leffler function and obtain the results on majorization and Fekete-Szegö problems. As for further research we could discuss sufficient and necessary conditions in relation to this subclass. Besides, linear combinations, distortion theory and another properties can be also explored.

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