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Building and Groups I

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Abstract: This is a pedagogical introduction to the theory of buildings of Jacques Tits and to some applications of this theory. This paper has 4 parts. In the first part we discuss incidence geometry, Coxeter systems and give two definitions of buildings. We study in the second part the spherical and affine buildings of Chevalley groups. In the third part we deal with Bruhat-Tits theory of reductive groups over local fields. Finally we discuss the construction of the p-adic flag manifolds.

Key words: Buildings; Incidence geometry; Coxeter groups; Chevalley groups; Reductive groups; Hecke algebras; P-adic symmetric spaces

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§ Introduction

This is a short presentation on the theory of "buildings" as an object of study in algebra and geometry. A building is a combinatorial geometry which is constructed from a Coxeter group.

Theory of buildings is one of the spectacular achievements in the second half of the twentieth century created by almost one person - Jacques Tits - for this work he was given the Abel Prize in 2008.

Tits first thought of buildings as an incidence geometry - see [Bruy16], [Uber11] for an introduction to this geometry. Likewise we begin our description of buildings as such a geometry - in fact we start with the definition of a graph - this is a combinatorial object we see everywhere.

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Then we explain what is a 'geometry' and what is an incidence geometry by using the projective plane as an example. Next we define a reflection as a linear map in a finite dimensional real vector space and introduce the concept of a finite reflection group which is a crucial ingredient in the theory of building. Finally we shall give two definitions of a building : one as a chamber system and the other as a chamber complex. We hope that this material is accessible to an undergraduate student who has taken courses in linear algebra and elementary group theory.

The study of buildings is part of geometric combinatorics (see [Buek95], [GGL95]). Because of its combinatorial nature this theory have many applications in modern engineering such as circuit design, power supply grids, neural networks, communication systems, drugs design. As theory of buildings is almost unknown in mathematics department in China, it is our goal here is to sketch the basic foundational materials for Chinese students who are interested in the applications of buildings theory.

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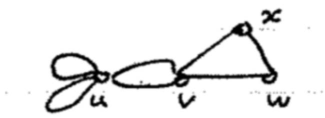
§1. Geometry

Instead of starting with a general definition of a building we begin with a simple revision in geometry.

1.1 Graphs A **graph** Δ is given by a pair $(V(\Delta), E(\Delta))$ together with a map v from $E(\Delta)$ to the set of unordered pairs of $V(\Delta)$. An element of $V(\Delta)$ is called a **vertex**, while an element of $E(\Delta)$ is called an **edge**. We also say Δ is a graph on $V(\Delta)$. For $e \in E(\Delta)$, $v(e) = \{v, w\}$ is taken to mean that e is a "path" joining the vertices v and w. Here is an example of a graph



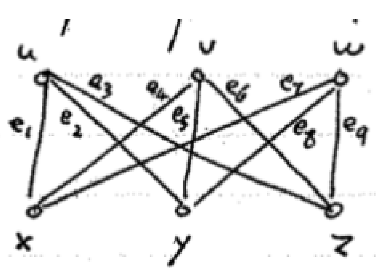
Clearly we can identify $E(\Delta)$ with a set of subsets of size 2 of $V(\Delta)$ and define a graph as a pair $(V(\Delta), E(\Delta))$ where $E(\Delta)$ is just a set of subsets of size 2 of $V(\Delta)$. If $E(\Delta)$ is the set of all two-elements subsets of $V(\Delta)$, then we call Δ a **complete graph**.

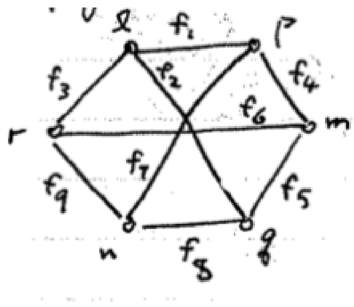
A graph isomorphism φ from a graph $G_1 = (V(G_1), E(G_1))$ to a graph $G_2 = (V(G_2), E(G_2))$ is given by bijections

$$\begin{cases} V(G_1) \leftrightarrow V(G_2) \\ E(G_1) \leftrightarrow E(G_2) \end{cases}$$

such that the edges joining any two vertices of G_1 are mapped to the edges joining the corresponding vertices of G_2.

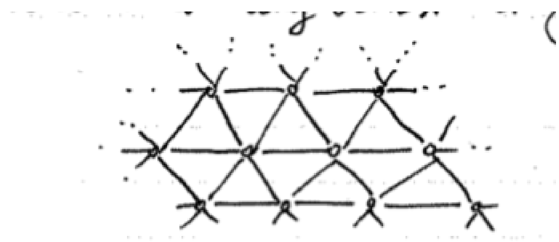
Example: the following graphs





are isomorphic, the required bijections are
Vertices: $u \leftrightarrow l, v \leftrightarrow m, w \leftrightarrow n, x \leftrightarrow p, y \leftrightarrow q, z \leftrightarrow r$.
Edges: $e_i \leftrightarrow f_i$.

A graph Δ is said to be *finite* if both $V(\Delta)$ and $E(\Delta)$ an finite set. A graph is said to be *locally finite* if the number of edges incident to any vertex is finite; e.g



1.2 Trees A *path* of length m in a graph $\Delta = (V, E)$ is a sequence

$$(x_0, x_1, \ldots, x_m)$$

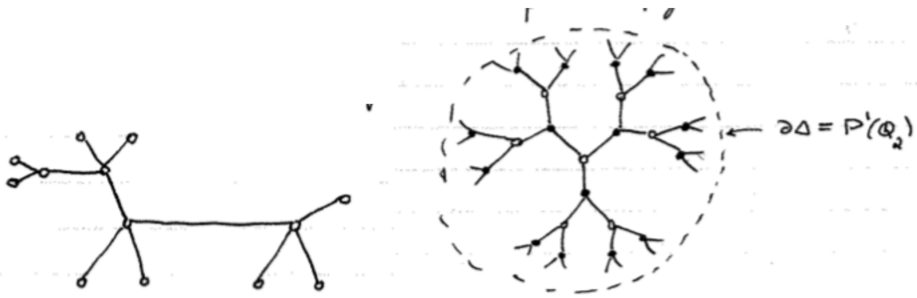
of vertices $x_0, x_1, \ldots, x_m$ such that $(x_{i-1}, x_i) \in E$ for all $1 \le i \le m$ and x_i is different from x_{i-2} for all $2 \le i \le m$. The distance dist(x, y) from x to y is the length of the shortest path from x to y. The **diameter** of a graph $\Delta = (V, E)$ is the supremum of the set

$$\{\mathrm{dist(u,v)} : \mathrm{u, v \in V}\}.$$

A *circuit* is a path of positive length whose first and last vertices are the same. The **girth** of a graph is the length of a shortest circuit (or ∞ if there are no circuits).

A graph G is said to be *connected* if given any pair of vertices v, w of G ,there is a path from v to w.

A **tree** is a connected non-empty graph without circuits. Here are two examples of trees - the picture on the right is a building with its boundary which we will explain in the next chapter.



The number of edges to which a vertex x of a tree belongs is called the valency of x. If every vertex has the same valency we say that this tree is homogeneous. Given two integers q, q_+ such that $1 \leq q < q_+$, we say a tree G is a semi-homogeneous tree of type (q, q_+) if every vertex has valency $q+1$ or $q_+ + 1$ and two adjacent vertices have different valencies. A homogeneous trees comes from the group $SL(2)$ over a local field and a semi-homogeneous tree comes from the group $SU(2,1)$ over a local field; these are the only two rank 1 quasi-split groups.

On trees there is an analogue of the theory of Laplacian operators and Poisson kernels on differentiable manifolds (see [KKM] for real manifolds; [FN91], [Ger80], [Ger84], [Kora], [Par06] for homogeneous trees and [GL002] for semi-homogeneous trees.)

It is rather surprising that such a simple object as a tree would have so much beautiful mathematics that Fields medallist Serre would write a book ([Serr80]) about it and Abel prize winner Langlands wrote a book ([Lan80]) using it. It is even more surprising that we can generalize the construction of trees to higher dimensions, namely to *buildings* !

1.3 Euclidean geometry At this point we ask : what is geometry? We cannot answer this question. But we can give an example of *a geometry*, namely the euclidean plane geometry. Most of our students go through high school doing hundreds of very difficult exercises with complicated diagrams in this geometry without asking what is this geometry - perhaps because this question is never in any examination and hence unimportant. But in order to understand modern mathematics this is indeed an important question. Let us give you a sketch of an answer as given by Euclid himself !

Around 300 B.C., Euclid of Alexandria laid an axiomatic foundation for geometry in his thirteen books called the *Elements*. There he proposed certain postulates, which were to be assumed without proof and then the geometry is the theorems which are deduced by logic from these postulates.

Euclid begins with 23 definitions of such terms as point, line, plane, circle, angle; in particular we give definitions 10 and 23:

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a *perpendicular* to that on which it stands.

23. *Parallel straight lines* are straight lines which, being in the same plane and being

produced indefinitely in both directions, do not meet one another in either direction.

Then Euclid says - let the following be postulated :

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The most famous is the postulate 5 which is called the "parallel postulate". Modern non-euclidean geometry was developed in respond to this axiom. For more details on Euclide's original book see - T. Heath, The thirteen books of Euclid's Elements, Dover Publications, New York, 1956.

Euclid's postulates have been corrected slightly over the years, since his original list was not quite complete and had a few logical flaws. Many mathematicians were involved in this process (Pasch, Peano, Veblen, Hilbert, etc.). For a modern treatment see Hilbert's famous geometry book Grundlagen der Geometrie of 1913. In such an axiomatic approach to geometry terms like point, line, contains, between, and congruent remain undefined and one assumes the postulates as given and develops geometry by deducing theorems from these postulates. Such method is the basis of machine proving in computer geometry.

1.4 Incidence geometry

[1] *Point-line geometry.*

Though incidence geometry does not appear explicitly in our discussions on buildings, it is still worthwhile to keep it in mind to serve as examples. Instead of giving a full definition of an incidence geometry (see [Uber11] p.3) we describe a simple case.

A point-line geometry is a triple $G = (P, L, \iota)$, with

(1) P a nonempty set, whose elements are called points,

(2) L a possibly empty set, disjoint from P, whose elements are called lines,

(3) ι a subset of $P \times L$, called the **incidence** relation, such that for every $\ell \in L$ there are at least two $x \in P$ for which $(x, \ell) \in \iota$.

Say two points x_1 and x_2 are called *collinear* if there is some line incident with x_1 and x_2.

A graph G is a point-line geometry such that

(1) every two distinct points of G are incident with at most one line,

(2) all lines are incident with precisely two points.

Let us move on to a more geometric situation.

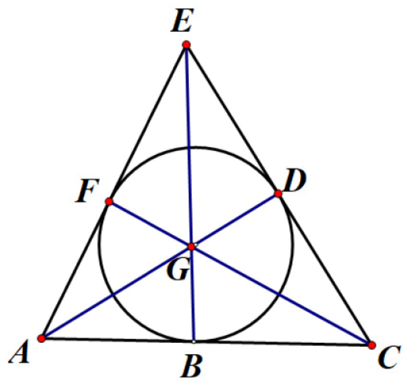
A **projective plane** S is a point-line geometry S such that

(1) every two distinct points are incident with precisely one line,

(2) for any two distinct lines of S, there is a unique point incident with both;

(3) there exist three noncollinear points.

Theorem 1.1 Every projective plane P with finite number of points such that every line of P is incident with exactly $2+1$ points is of the form



(See [Uber11] Thm 8.7 p.56).

The projective plane P has :

• seven points $\{A, B, C, D, E, F, G\}$,

• seven lines $\{A, B, C\}$, $\{A, G, D\}$, $\{A, F, E\}$, $\{B, G, E\}$, $\{B, D, F\}$, $\{C, G, F\}$, $\{C, D, E\}$ (the inscribed circle $\{B, D, F\}$ is counted as a line);

• every line is incident with exactly 3 points as indicated and

• every pair of points determine a line.

[2] *Types.*

We extend the point-line geometry above.

Definition 1.2 An **incidence geometry** over a set I is the data $\Gamma = (S, S \xrightarrow{\tau} I, *)$ where S is a set, $*$ is a binary reflexive relation on S such that for $\tau a = \tau b$, $a = b \Leftrightarrow a * b$.

We call τa the type of a, $*$ the incidence and the cardinality $|I|$ the rank of the geometry.

A flag F of Γ is a set of pairwise incident elements of S. Say F is of type $\tau(F)$. Two flags are said to be incident if their union is a flag. Let T be the set of all $a \in S - F$ such that a is incident to F. Suppose $\tau(F) = J \subset I$. Then the geometry $(T, \tau|_T : T \to I - J, * \cap (T \times T))$ over $I - J$ is called the residue of F in Γ and denoted by Res F of Γ_F. The corank of F is $|I - J|$.

The graph of $\Gamma = (S, S \xrightarrow{\tau} I, *)$ has vertices the set S and edges join the incident pairs. We say Γ is connected if its graph is connected and Γ is residually connected if the residue of every flag of corank ≥ 2 (of corank 1) is connected (nonempty).

For $m \in \cup\{\infty\}$ and $m \geq 2$, a geometry of rank 2 is called a generalized m-gon if its graph has diameter m and girth $2m$ and if every vertex of the graph belongs to at least two edges.

Let us take a finite set I. A *Coxeter matrix* on I is an array $M = [m_{ij}]_{i,j \in I}$ with such that each m_{ij} is either a positive integer or the symbol ∞, $m_{ij} = m_{ji} \geqslant 2$ if $i \neq j$ and $m_{ii} = 1$.

Definition 1.3 An incidence geometry over I of type M is a residually connected geometry over I such that for $i, j \in I$ and $i \neq j$, the residue of any flag of type $I - \{i, j\}$ is a generalized m_{ij}-gon. [Tit81] §1.4.

1.5 Projective space[1] *Projective n space.*

For most of us our idea of a space was developed from the moment we open our eyes to this world. To a student of mathematics since the times of Descartes we know this space as ^{3} with a metric measuring the distance between two points $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

It is natural to extend this space by replacing the number 3 by any positive integer n to obtain an n-dimensional euclidean space and even by infinity to obtain a Hilbert space. We can also replace the field of real numbers by any field k that we study in an algebra course.

But the next idea : that of a projective space is really very different kind of space. Let us first give a definition.

Let k be a field and k^{n+1} denotes the $n + 1$ dimensional vector space over k. A vector or a point in k^{n+1} is of the form $x = (x_0, x_1, x_2, \ldots, x_n)$. For $x, y \in k^{n+1}$ such that $x \neq (0, 0, \ldots, 0) \neq y$ we define a relation $\sim$ by : $x \sim y$ if there is a $\lambda \in k$ and $\lambda \neq 0$ such that $x_i = \lambda y_i$ for all $0 \leq i \leq n$. It is easy to check this is an equivalence relation on $k^{n+1} \setminus (0, 0, \ldots, 0)$ and the equivalence containing $x \neq 0$ is the line passing through 0 and x. The set of all equivalence classes is called the **projective n-space** over the field k and is denoted by $^n(k)$.

We can repeat this construction replacing k^{n+1} by any $n + 1$ dimensional vector space V to obtain the projective space of V which will be denoted by V. A 'point' of V is then a 1 dimensional subspace of V.

To learn more about the projective space we refer you to Coxeter [Coxe73], Artin [ArtE55] & [ArtE57], Mumford [Mumf81], Griffiths and Harris [GH78].

It is said that the projective plane is implicit in the work of the ancient Greek Pappus. Then around the time of the Renaissance in Europe the idea of projective space comes up in the study of perspective in painting. The first formalization known is due to G. Desargues, with the book Brouillon Projet d'une atteinte aux événements des rencontres du Cône avec un Plan (Rough draft for an essay on the results of taking plane sections of a cone) published in 1639. Finally it is 19th century Poncelet introduce the line at infinity, Möbius introduced the homogeneous coordinates, Steiner gave the first axiomatic synthetic treatment and Karl von Staudt was the first to adopt a fully rigorous approach in his book "Geometrie der Lage" (1847).

[2] *Projective plane.*

To see that the point-line geometry P given in the theorem 1.1 is indeed the "usual" projective plane ^{2} we shall show that P is the projective plane $\mathbb{P}^2(\mathbb{F}_2)$ over the field $\mathbb{F}_2 = \{0, 1\}$ of 2 elements. We proceed as follows -

(1) Recall that $\mathbb{P}^2(\mathbb{F}_2)$ can be defined as the set $\mathbb{F}_2^3 \backslash \{0, 0, 0\}$ modulo $\sim$. Here the relation $\sim$ is defined by : $(x, y, z) \sim (x', y', z')$ if there is a $\lambda \in \mathbb{F}_2$ and $\lambda \neq 0$ such that $(x, y, z) = \lambda (x', y', z')$. As $\mathbb{F}_2 = \{0, 1\}$, so λ can only be $= 1$. Thus $\mathbb{P}^2(\mathbb{F}_2) = \mathbb{F}_2^3 \backslash \{0, 0, 0\}$ and so $\mathbb{P}^2(\mathbb{F}_2)$ has 7 elements.

(2) Each point in $\mathbb{P}^2(\mathbb{F}_2)$ is a line in $\mathbb{F}_2^3$ and every pair of points in $\mathbb{P}^2(\mathbb{F}_2)$ determines a plane in $\mathbb{F}_2^3$. Each such plane is given by an equation $ax + by + cz = 0$. Thus the number of planes is just the number of (a, b, c) in $\mathbb{F}_2^3$ modulo $\sim$, i.e. it is the number of elements of $\mathbb{P}^2(\mathbb{F}_2)$. Hence we see that every pair of points in $\mathbb{P}^2(\mathbb{F}_2)$ determines a line in $\mathbb{P}^2(\mathbb{F}_2)$ which has a total of 7 lines. (This is just duality.) Moreover every plane in $\mathbb{F}_2^3$ is a two dimensional vector space over $\mathbb{F}_2$ and so it has 3 nonzero elements. This means that each line in $\mathbb{P}^2(\mathbb{F}_2)$ has 3 points.

What is most spectacular about this example is that there is a group action.

Let X be a set and G be a group with identity element 1_G. We say that G acts on X if there is a map $\alpha : G \times X \to X$ such that
(1) $\forall\, x \in X$, $\alpha(1_G, x) = x$;
(2) $\forall\, x \in X$, $\forall\, g_1, g_2 \in G$, $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$. We call the map α an action of G on X.

For example we can take X to be the set of all real numbers and G to be the additive group of all integers and for an integer g and a real number x the action is

$$\alpha(g, x) := g + x$$

This most important action is the basis of Fourier analysis.

Next let k be a field and $SL_3(k)$ denotes the group of 3×3 matrices of determinant 1 with coefficients in k. For $A, B \in SL_3(k)$ we write $A \sim B$ if there is a nonzero $\lambda \in k$ such that $A = \lambda B$. This defines an equivalence relation and the set $PSL_3(k)$ of equivalence classes is still a group under matrix multiplication (check !).

Take $[A] \in PSL_3(k)$ and we define an action of $PSL_3(k)$ on ${}^2(k)$ by matrix multiplication :

$$\alpha([A], [x]) := [Ax], \quad \text{where } [A] \in PSL_3(k), [x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ x_i \in k.$$

In the case of our projective plane ${}^2({}_2)$ this says the *simple group* $PSL_3({}_2)$ of order 168 acts on ${}^2({}_2)$.

A geometric structure with a group action is indeed the turning point in the history of geometry after Euclid !

In 1872 Felix Klein published the Erlangen program. One of the themes of this program says that given a geometric structure S there is a group G such that the properties of S can be characterized as invariants of G. For example if S is the projective space ^{n} then we take $G = PSL(n)$. Conversely given a group G Tits constructs a geometric structure S on which G acts as transformations preserving its geometric properties; Tits called S a building.

[3] *Projective space as an incident geometry.*

Definition 1.4 A *projective geometry* of dimension n is the data $PG_n = (S, S \xrightarrow{rk} \{1, 2, \ldots, n\}, *)$ where S is a set, $*$ is a relation on S and the following conditions hold.
PG1) If $a * b$, $b * c$ and $rk(a) \le rk(b) \le rk(c)$, then $a * c$.
PG2) If $rk(a) = 1$, $rk(b) = j < n$ and $a \not* b$ then there exists c satisfying

i) $rk(c) = j + 1$
ii) for every d one has $a * d$ and $b * d$ if and only if $c * d$ and $rk(d) \geq j + 1$.
PG3) If $rk(a) = n$, $rk(b) = j > 1$ and $a \not* b$ then there exists c satisfying
i) $rk(c) = j - 1$
ii) for every d one has $d * a$ and $d * b$ if and only if $d * c$ and $rk(d) \leq j - 1$.
PG4) If $rk(a) = 2$ then there exists at least 3 objects b_1, b_2, b_3 such that $rk(b_i) = 1$ and $b_i * a$.

A n dimensional projective geometry is an incident geometry over the set $\{1, 2, \ldots, n\}$.

If V is an $n + 1$ dimensional vector space over a division ring, we let S to be the set of all non-empty proper subspaces of V, rk is just the dimension and $*$ is the inclusion $\subseteq$. Then this is an n dimensional projective geometry; conversely we have the following theorem.

Theorem 1.5 Let $n \geq 3$. Then any n dimensional projective geometry is of the form V for some $n + 1$ dimensional vector space V over some division ring.

This is called the existence of coordinates theorem for projective geometry; (see [Tit84] Thm I.1.1 p.2; [Sei06] Chap 9 §1 p.166) this should be distinguished from the fundamental theorem for projectivities in projective geometry (see [Sei06] Chap 1 §14 p.25). A 2 dimensional projective geometry is called a projective plane. Not all projective planes come from vector spaces; those which do not are called non-Desarguesian.

Theorem 1.6 An incidence geometry of type A_n is an n-dimensional projective geometry.

See [Tit84] Thm I.1.1 p.6; [Tit81] §6.1.5, Prop 6 p. 540. The Coxeter matrix A_n is $n \times n$ matrix (m_{ij}) with $m_{ii} = 1$, $m_{12} = m_{n,n-1} = 3$, $m_{i,i-1} = m_{i,i+1} = 3$ for $2 \leq i \leq n - 1$ and the rest of entries $= 2$. For example

$$A_4 = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}$$

This theorem marks the beginning of the theory of buildings - see [Tit62],[Tit81].

Before we leave let us point you to two books on incidence geometry - [Bruy16], [Uber11]; for example you can try to read Chapter 1 of [Uber11].

§2. Preliminaries

We need to take a digression. Coxeter groups are the basic ingredients in the construction of buildings. Our references are [Bour81], [Hum90] and [Hum72].

2.1 Coxeter system Let I be a set. A **Coxeter matrix** is an array $M = [m_{ij}]_{i,j \in I}$ with such that each m_{ij} is either a positive integer or the symbol ∞, $m_{ij} = m_{ji} \geqslant 2$ if $i \neq j$ and $m_{ii} = 1$.

The **Coxeter graph** (or diagram) of a Coxeter matrix $[m_{ij}]$ is the undirected graph Π with vertex set I and edge set consisting of all unordered pairs $\{i, j\}$ such that $m_{ij} \geq 3$ (including

$m_{ij}=\infty$) together with the labeling which assigns the label m_{ij} to each edge $\{i,j\}$. This means that we join vertices i and j by an edge labelled m_{ij} whenever this number (∞ allowed) is at least 3. If distinct vertices i and j are not joined, it is then understood that $m_{ij}=2$. As a simplifying convention, the label $m_{ij}=3$ may be omitted. A Coxeter graph is called *irreducible* if its underlying graph is connected. The *rank* of a Coxeter graph is the cardinality of its vertex set I.

Given a Coxeter graph Π with Coxeter matrix M, the **Coxeter group** of type Π (or M) is the group W having a set of generators $S=\{w_i : i\in I\}$, subject only to relations of the form

$$(w_iw_j)^{m_{ij}}=1,\ \ i,j\in I,\ m_{ij}\neq\infty$$

where the array $M=[m_{ij}]_{i,j\in I}$ is the given Coxeter matrix.

Let $f\mapsto w_f$ denote the unique extension of the map $i\mapsto w_i$ to a homomorphism from the free monoid with generators I to W ($w_\emptyset=1$). ([Wei03] Def 2.2 p.9)

The pair (W,M) is called a **Coxeter system** of type Π (In [Bour81] and [Hum90] the pair (W,S) is called a Coxeter system).

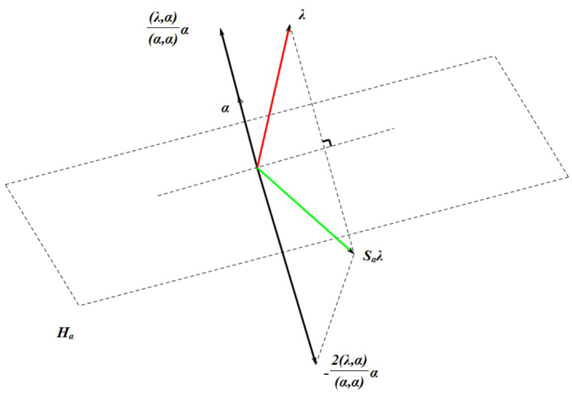
2.2 Finite reflection group By a real euclidean space we mean a finite dimensional vector space V over the field of real numbers endowed with a positive definite symmetric bilinear form $(\bullet,\bullet)$. A **reflection** is a linear map $s:V\to V$ which sends some nonzero vector α to $-\alpha$ while fixing pointwise the hyperplane

$$H_\alpha=\{\lambda\in V:(\lambda,\alpha)=0\}$$

orthogonal to α. Let us write s_α for this s. Then

$$s_\alpha\lambda=\lambda-\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha.$$

This is explained by the following diagram in which we identify V with its dual space V^* using the given bilinear form $(\bullet,\bullet)$ so that the hyperplane H_α appears as the plane perpendicular to α. Then $\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$ is the projection of λ onto the line containing the vector α and thus $s_\alpha\lambda$ as given by the above formula is the 'mirror' image of λ obtained by reflecting λ using H_α as the 'mirror' (= reflecting plane).



A **finite reflection group** is a finite group generated by reflections; it is a finite subgroup of of the orthogonal group $O(V, (\bullet, \bullet))$.

Theorem 2.1 A Coxeter group W is finite if and only if W is a finite reflection group. ([Hum90] §6.4, p.133)

By a **root system** we mean a subset Φ of a real euclidean space V satisfying the conditions:
(R0) Φ is finite, spans V, and does not contain 0.
(Rl) $\Phi \cap \alpha = \{\alpha, -\alpha\}$, $\forall\, \alpha \in \Phi$.
(R2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
(R3) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in$ for all $\alpha, \beta \in \Phi$.
([Hum72] §9.2, p.42; it is also called a crystallographic root system in [Hum90] §2.9, p.39) The group $W(\Phi)$ generated by all reflections s_α ($\alpha \in \Phi$) is known as the **Weyl group** of Φ. W is finite ([Hum72] §9.2, p.43) and so it is a finite Coxeter group. We sometimes call a root system a Lie root system.

Theorem 2.2 If Φ is a root system in V then there is a subset Υ of Φ (called a base) such that
(B1) Υ is a basis of V,
(B2) each root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Upsilon} m_\alpha \alpha$ with integers m_α and all m_α are nonnegative or all nonpositive. ([Hum90] §10.1, p.48) This means that $\Phi = \Phi^+ \sqcup -\Phi^+$, where Φ^+ consists of those roots $\beta = \sum_{\alpha \in \Upsilon} m_\alpha \alpha$ with $m_\alpha \geq 0$. Φ^+ is called the set of positive roots. An element in Υ is called a **simple root**.

Choose an ordering of the simple roots $\Upsilon = \{\alpha_1, \ldots \alpha_\ell\}$. Set $c_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$. The integer $c_{ij}c_{ji}$ takes only the values $0, 1, 2, 3$ ([Hum72] §9.4). Define the **Dynkin diagram** as the graph with simple roots as vertices, α_i is joined to α_j by $c_{ij}c_{ji}$ edges, and whenever a double or triple edge occurs, we add an arrow pointing to the shorter of the two roots.

Say a root system is irreducible if it cannot be partitioned into two proper, orthogonal subsets. Call the number ℓ of elements in a base Υ of Φ the rank of Φ.

Theorem 2.3 If Φ is an irreducible Lie root system of rank ℓ, its Dynkin diagram is one of the following (ℓ vertices in each case):

A_ℓ $\circ_1$ —— $\circ_2$ —— $\circ_3 \cdots \circ_{\ell-1}$ —— $\circ_\ell$

B_ℓ $\circ_1$ —— $\circ_2$ —— $\circ_3 \cdots \circ_{\ell-1} \Longrightarrow \circ_\ell$

C_ℓ $\circ_1$ —— $\circ_2$ —— $\circ_3 \cdots \circ_{\ell-1} \Longleftarrow \circ_\ell$

D_ℓ $\circ_{\ell-1}$

$\circ_1$ —— $\circ_2 \cdots \circ_{\ell-3}$ —— $\circ_{\ell-2}$

$\circ_\ell$

E_6

$$\begin{array}{ccccccccc} & & & & \circ_2 & & & & \\ & & & & | & & & & \\ \circ_1 & — & \circ_3 & — & \circ_4 & — & \circ_5 & — & \circ_6 \end{array}$$

E_7

$$\begin{array}{ccccccccccc} & & & & \circ_2 & & & & & & \\ & & & & | & & & & & & \\ \circ_1 & — & \circ_3 & — & \circ_4 & — & \circ_5 & — & \circ_6 & — & \circ_7 \end{array}$$

E_8

$$\begin{array}{ccccccccccccc} & & & & \circ_2 & & & & & & & & \\ & & & & | & & & & & & & & \\ \circ_1 & — & \circ_3 & — & \circ_4 & — & \circ_5 & — & \circ_6 & — & \circ_7 & — & \circ_8 \end{array}$$

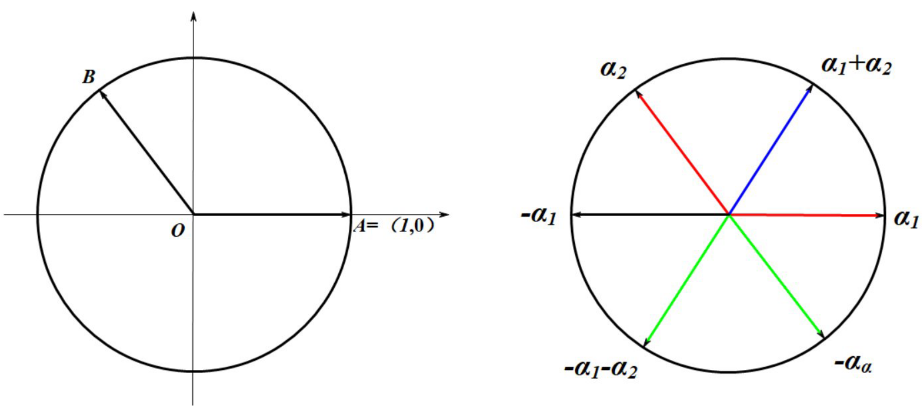
F_4 $\quad \circ_1 — \circ_2 \Rrightarrow \circ_3 — \circ_4$

G_2 $\quad \circ_1 \Lleftarrow \circ_2$

Note : the Coxeter graph of a finite irreducible Coxeter group are different, see - [Bour81] Thm 1 no. 4.1, p.193.

To give some feelings about root system we construct an example - the root system of type A_2. In the diagram below on the left hand side we have a unit circle and the angle $\measuredangle AOB = 2\pi/3$. On the right the vector α_1 (resp. α_2) is OA (resp. OB.)

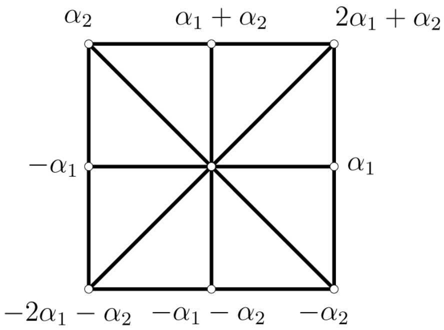
A_2



The simple roots are $\Upsilon = \{\alpha_1, \alpha_2\}$. The positive roots are $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The root system A_2 is

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}.$$

In the next diagram we show a root system of type C_2. In this diagram α_1 and $\alpha_1 + \alpha_2$ forms the sides of a unit square.



The simple roots are $\Upsilon = \{\alpha_1, \alpha_2\}$. The positive roots are $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$. The root system A_2 is

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2)\}.$$

There are exactly 3 types of irreducible root systems in the plane, they are A_2, C_2 and G_2.

The root system of type G_2 is :

$$\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2), \pm(\alpha_1 + 3\alpha_2) \pm (2\alpha_1 + 3\alpha_2)\}.$$

Perhaps at this point it may be good to know a bit more about root systems. If you are familiar with root systems you will find it easier to get comfortable with buildings. Chapter III of Humphreys [Hum72] has a self contained account, you do not have read chapters I and II, just go to chapter III. And if you have more time you can also try [Hum90] Chapters 1,4,5.

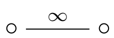
2.3 Affine reflection group

Given a Coxeter graph Π of finite rank with Coxeter matrix M. Then the symmetric matrix M defines a bilinear form x^tMy $(x, y \in^n)$. We say M (or Π) is positive definite if $x^tMx > 0$ for all $x \neq 0$, positive semi-definite if $x^tMx \geq 0$ for all x.

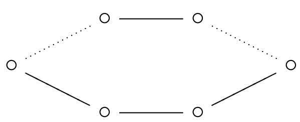
Theorem 2.4 A connected Coxeter graph of finite rank which is positive semidefinite but not positive definite is isomorphic to one of the following graphs.

No two of the above graphs are isomorphic. ([Bour81] no 4.3 Thm 4 p.199; [Hum90] p.34, 96, 152)

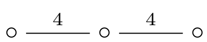
$\widetilde{A}_1$



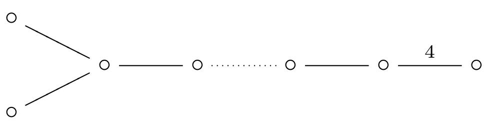
$\widetilde{A}_\ell(\ell \geq 2, \ell + 1$ vertices)



$\widetilde{B}_2$



$\widetilde{B}_\ell(\ell \geq 3, \ell + 1$ vertices)



$\widetilde{C}_\ell(\ell \geq 3, \ell+1$

$\widetilde{D}_\ell(\ell \geq 4, \ell+1$ vertices)

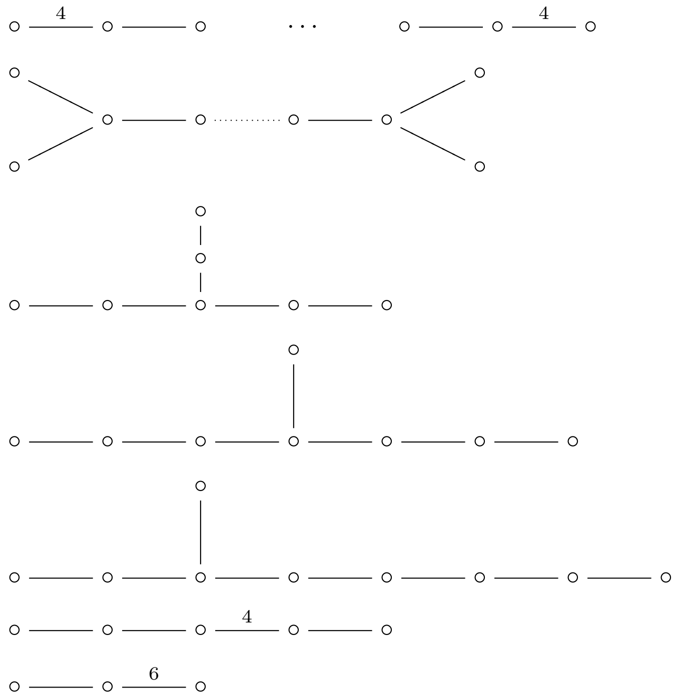
$\widetilde{E}_6$

$\widetilde{E}_7$

$\widetilde{E}_8$

$\widetilde{F}_4$

$\widetilde{G}_2$



Let V be a a real euclidean space. We let $T(V)$ be the group of all translations on V; an element of $T(V)$ is of the form

$$t(\lambda) : \mu \mapsto \lambda + \mu, \quad \lambda, \mu \in V.$$

For a linear isomorphism $g \in GL(V)$, we have $gt(\lambda)g^{-1} = t(g\lambda)$. Thus we can introduce the affine group Aff(V) := GL(V) ⋉ T(V) as a semidirect product. An reflection relative to a hyperplane which do not necessarily pass through the origin is in Aff(V). For a formal treatment of **affine spaces** and affine transformation see [Port69] Chap 4.

Now we take a Lie root system Φ in a real euclidean space V. Introduce the notation $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

A root $\alpha \in \Phi$ and an integer k define an affine hyperplane

$$H_{\alpha,k} := \{\lambda \in V : (\lambda, \alpha) = k\}.$$

Note that $H_{\alpha,k}$ is $H_{\alpha,0}$ translated by $\frac{k}{2}\alpha^\vee$. Define the corresponding affine reflection with respect to $H_{\alpha,k}$ as follows:

$$s_{\alpha,k}\lambda = \lambda - ((\lambda, \alpha) - k)\alpha^\vee.$$

Clearly $s_{\alpha,k}$ fixes $H_{\alpha,k}$ pointwise.

We define the **affine Weyl group** $W_{\text{aff}}(\Phi)$ to be the subgroup of Aff(V) generated by all affine reflections $s_{\alpha,k}$, where $\alpha \in \Phi$, $k \in$.

Let us assume that Φ is irreducible, then there is a unique *highest root* $\tilde{\alpha}$ (which is long if there are two root lengths), this means that for all positive roots β, $\tilde{\alpha} - \beta$ is a nonnegative

-linear combination of simple roots ([Bour81] Chap VI, no 1.8 Prop 25). Let be the set of all connected components of $V \setminus \bigcup_{\alpha\in\Phi,k\in} H_{\alpha,k}$. Each element of is called a *chamber* or an **alcove**. For example

$$_0 = \{\lambda \in V : (\lambda, \tilde{\alpha}) < 1 \;\&\; 0 < (\lambda, \alpha) \;\forall\; \alpha \in \Upsilon\}$$

is an alcove.

Theorem 2.5 Given a base Υ of an irreducible Lie root system Φ. Let $S = \{s_{\alpha,0} : \alpha \in \Upsilon\} \cup \{s_{\tilde{\alpha},1}\}$. Then the affine Weyl group $\widetilde{W}(\Phi)$ is a Coxeter group having S as a set of generators. Moreover the Coxeter graph of this Coxeter group is one of those in theorem 2.4.

([Hum90] §4.7 p.95; §6.5, p.133; [Bour81] Chap VI §4.3)

We define an **affine Coxeter group** as a Coxeter group associated with a Coxeter graph in theorem 2.4 or a direct product of such groups ([Wei09] p. 1).

§3. Chamber Systems

We hope that the preceding paragraphs has given you a warm-up exercise to prepare you for the next set of definitions/axioms.

We give a definition of a building as a chamber system.

3.1 Edge-colored graphs An edge-colored graph is a graph $\Delta = (V, E)$ endowed with a surjective map $E \to I$. We think of elements of I as colors; but we call I the *index* set of Δ.

A change of terminology. Suppose $\Delta = (V, E)$ is an edge-colored graph, we sometimes refer to the elements of V as **chambers** rather than vertices and we will write Δ in place of V. We write $x \sim_i y$ for the statement '$\{x, y\}$ is in E and has color i' and we say x and y are i-adjacent.

Given $x, y \in \Delta$. A **gallery** of length k from x to y is a sequence $\gamma = (x = u_0, \ldots, u_k = y)$ of $k + 1$ chambers such that

$$u_{j-1} \sim_{i_j} u_j,\ 1 \leq j \leq k$$

and the type of γ is the word $i_1 \cdots i_k$ (in the free monoid generated by I). For $J \subseteq I$, a J-gallery is a gallery whose type is in the free monoid generated by J. We say Δ is connected (resp. J-connected) if for any two chambers x, y there exists a gallery (resp. a J-gallery) from x to y.

Suppose that Δ is a connected edge-colored graph with index set I and let J be a subset of I. A J-**residue** of Δ is a connected component of the graph obtained from Δ by deleting all the edges whose color is not in J (but without deleting any vertices). The type of a residue (including Δ itself) is the set of colors that appear on its edges (i.e. its index set) and the **rank** of a residue is the cardinality of its type. A i-**panel** is a residue of rank 1 of type $\{i\}$.

A **chamber system** (Δ, I) is a connected edge-colored graph Δ with index set I such that all the panels are complete graphs having at least two chambers. A chamber system is called **thick** (respectively, **thin**) if all its panels contain at least three chambers (respectively, exactly two chambers).

A homomorphism of chamber system is $(\Delta \xrightarrow{\rho} \Delta'; I \xrightarrow{\sigma} I')$ such that for all $i \in I$, ρ takes each i-panel of Δ to a $\sigma(i)$-panel of Δ'. We say ρ is special if $I = I'$ and σ is the identity map.

Example. Let Π be a Coxeter graph with vertex set I and let (W, r) be the Coxeter system of type Π. We define Σ_Π to be the edge-colored graph with the elements of W as chambers and we set

$$x \sim_i y \text{ for } i \in I \Leftrightarrow x^{-1}y = r_i.$$

Since $r_i \neq 1$, $x \sim_i y \Rightarrow x \neq y$. $r_i^2 = 1 \Rightarrow$ the relation $\sim_i$ is symmetric. Since $r_i \neq r_j$ whenever $i \neq j$, the color of an edge is well defined. Σ_Π is a thin chamber system, i.e. each chamber is i-adjacent to exactly one chamber for each color $i \in I$. Moreover for any $x \in \Sigma_\Pi$ and any f in the free monoid generated by I, there is a unique gallery of type f in Σ_Π beginning at x and ends in xr_f ([Wei03] Prop 2.5 p.12).

A **Coxeter chamber system** of type Π is an edge-colored graph isomorphic to Σ_Π.

3.2 Buildings

[1] Let M_I be the free monoid on I. For $i \neq j$ such that $m_{ij} \neq \infty$ put

$$p(i,j) = \begin{cases} (ij)^{m_{ij}/2} & \text{if } m_{ij} \text{ is even} \\ j(ij)^{(m_{ij}-1)/2} & \text{if } m_{ij} \text{ is odd} \end{cases}$$

An elementary homotopy is a transformation of a word of the form $fp(i,j)g$ into the word $fp(j,i)g$; two words f, g are homotopic if f can be transformed into g by a sequence of elementary homotopies. A word in M_I is called reduced (with respect to Π) if it is not homotopic (as defined in 4.1 of [37]) to a word of the form $fiig$ for some $i \in I$ and some $f, g \in M_I$.

[2] Let Π be a Coxeter graph with vertex set I and let (W, r) be the Coxeter system of type Π. A **building** of type Π is a pair (Δ, δ), where Δ is a chamber system whose index set is I and $\delta : \Delta \times \Delta \to W$ has the property that for each reduced word f in the free monoid generated by I and for each ordered pair (x, y) of chambers, $\delta(x, y) = r_f$ iff there is a gallery in Δ of type f from x to y. δ is called the Weyl distance function of Δ. ([Wei03] Def 7.1.)

The Coxeter chamber system Σ_Π is the only (up to special isomorphism) thin building of type Π ([Wei03] 8.11).

A **spherical building** is a building whose Coxeter group is finite; and an **affine building** is a building whose Coxeter group is affine.

Affine Coxeter groups can be characterized as groups generated by reflections of an affine space ([Bour81] Chap VI, §4.3).

[3] Let (Δ, δ), (Δ', δ') be two buildings of the same type (and thus having the same index set I and the same Coxeter group W). A map π from a subset X of Δ to Δ' will be called an isometry if for all $x, y \in X$ we have

$$\delta'(\pi x, \pi y) = \delta(x, y).$$

[4] Let Δ be a building of type Π and let Σ be a Coxeter chamber system of type Π. An **apartment** is the image of a special isomorphism from Σ into Δ. Apartments have the following

properties:
(1) Every isometry from a subset of Σ to Δ extends to a special isomorphism from Σ to Δ ([Wei03] 8.2,8.5).
(2) Every two chambers lie in an apartment ([Wei03] 8.6).
(3) Apartments are convex ([Wei03] 8.9).

[5] Let Δ be a chamber system whose index set is the vertex set I of the Coxeter diagram Π and let be a collection of subgraphs isomorphic to the Coxeter chamber system Σ_{Π}. Suppose that the following hold:
(1) For every two chambers x and y of Δ, there exists an element of that contains them both.
(2) For every two elements $A, A' \in$ and for every pair x, y of chambers in $A \cap A'$, there exists a special isomorphism $A \to A'$ that fixes x and y.
(3) For every two elements $A, A' \in$, for every chamber x in $A \cap A'$ and for every panel P such that $P \cap A$ and $P \cap A'$ are both non-empty, there exists a special isomorphism $A \to A'$ that fixes x and maps $P \cap A$ to $P \cap A'$.
Then Δ is a building of type Π ([Ron92] 3.11; [Wei09] 29.35).

§4. Chamber Complexes

We give in this section a definition of buildings as chamber complexes.

4.1 Complexes

[1] A **partially ordered set** is a set S together with a binary relation $a \leq b$ satisfying the following conditions:
1) $a \leq a$.
2) $a \leq b$ and $b \leq a$ implies that $a = b$.
3) $a \leq b$ and $b \leq c$ implies that $a \leq c$.

An element $s \in S$ is a lower bound of a subset A of S if $s \leq a$ for every $a \in A$. The element s is a greatest lower bound or inf of A if s is a lower bound of A and $t \leq s$ for every lower bound t of A. The greatest lower bound of $\{a, b\}$ is denoted by $a \wedge b$.

Example. Take any set T. Let S be the set of all subsets of T. For subsets A, B of T, we take the relation $A \leq B$ to mean $A \subseteq B$.

[2] Consider a set Θ endowed with partial ordering $\subseteq$. Θ is called a **simplex** if Θ is isomorphic to the set of all subsets of some set partially ordered by inclusion.

[3] We say that Θ is a **complex** if:
(a) $\forall\ A \in \Theta$, the set $\{B \in \Theta : B \subseteq A\}$ forms a simplex,
(b) $\forall\ A, B \in \Theta, \exists$ a greatest lower bound $A \wedge B$ in Θ. A complex Θ contains a unique minimal set which is denoted by 0.

[4] A subset Θ' of Θ with induced partial ordering is called a *subcomplex* if $\forall\ A \in \Theta', B \in \Theta$ with $B \subseteq A$ we have $B \in \Theta'$.

For $A \in \Theta$, we define the **star** of A as

$$\mathrm{StA} = \{\mathrm{B} \in \Theta : \mathrm{B} \supseteq \mathrm{A}\}.$$

[5] For $A \in \Theta$, we define the RankA to be the number of B such that B is minimal with respect to the properties $B \subseteq A, B \neq 0$.

Thus the set of elements B with $B \subseteq A$ is isomorphic to the set of all subsets of a set with cardinality RankA.

For $A \subseteq B$ the *codimension* is defined to be

$$\mathrm{codim_B A = rank_{StA} B.}$$

In particular

$$\begin{aligned}&\mathrm{codim_B A} = 0 \Leftrightarrow A = B,\\&\mathrm{codim_B A} = 1 \Leftrightarrow A \neq B \ \& \ \nexists X : A \subset X \subset B.\end{aligned}$$

4.2 Chamber Complex

[1] A complex Θ is called a **chamber complex** if

(a) every element of Θ is contained in a maximal element;

(b) given any 2 maximal elements C, C' of Θ $\exists$ a finite sequence

$$C = C_0, C_1, C_2, \ldots, C_m = C'$$

of elements of Θ such that

$$\mathrm{codim_{C_{i-1}}(C_{i-1} \cap C_i) = codim_{C_i}(C_{i-1} \cap C_i) \leqslant 1 \text{ for } i = 1, \cdots, m}$$

Each C_i is a maximal element.

An maximal element of a chamber complex will be called a **chamber**.

A sequence of chambers satisfying the conditions in (b) will be called a **gallery**.

Two chambers C, C' are said to be **adjacent** if $\mathrm{codim_C(C \cap C') = 1 = codim_{C'}(C \cap C')}$.

A chamber complex is said to be **thin** if every element of codimension 1 is contained in exactly 2 chambers. We say a chamber complex is **thick** if every element of codimension 1 is contained in at least 3 chambers.

[2] A map $\alpha : \Theta \to \Theta'$ is called a *morphism* of chamber complexes if
(a) $C \in \Theta$ is a chamber $\Rightarrow \alpha(C)$ is a chamber in Θ';
(b) for any chamber $C \in \Theta$, α induces an isomorphism between simplexes determined by C and $\alpha(C)$.

[3] Let Φ be a Lie root system in an euclidean space V; let $\Upsilon = \{\alpha_i : i \in I\}$ be a base of Φ and $W = W(\Phi)$ be the Weyl group of Φ.

For any subset $J \subset I$ let W_J be the subgroup of W generated by all s_{α_j} for $j \in J$.

Let $\Sigma(W(\Phi), \Upsilon)$ be the partially ordered set $\{wW_J : w \in W, J \subseteq I\}$ ordered by the opposite of inclusion relation, namely say $B \leqslant A$ exactly when $B \supseteq A$ as subsets of W, in which case, say B is a face of A.

Theorem 4.1 $\Sigma(W(\Phi), \Upsilon)$ is a thin chamber complex. The elements of W are chambers and w, ws_{α_i} are adjacent.

([Car72] 15.4, p.289.)

There is a more 'geometric' way to see $\Sigma(W(\Phi), \Upsilon)$.

We introduce an equivalence relation on V by setting $x \sim y$ if, for each hyperplane H_α, $\alpha \in \Phi$, the points x, y are either both in H_α, or both not in H_α but on the same side H_α. Let be the set of $\sim$-equivalence classes. has partial order $\preccurlyeq$ defined by $C_1 \preccurlyeq C_2$ if C_1 is contained in the closure of C_2.

For $w \in W$,w transforms each reflecting hyperplane into another, so for $C \in \mathcal{C}, w(C)$ is defined and is in $\mathcal{C}$. In his way W acts on $\mathcal{C}$.

The connected components of

$$V \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha$$

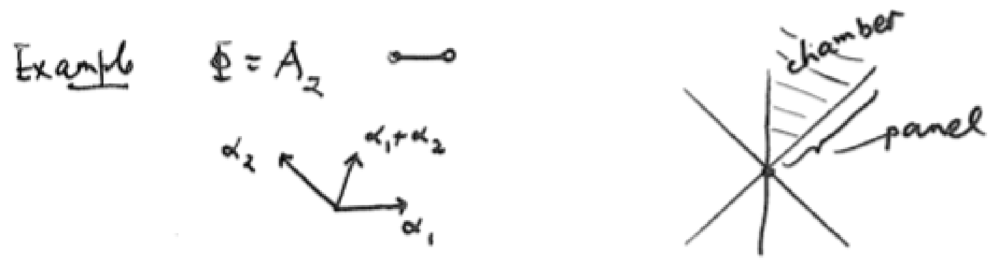
are called *chambers*.

$$_\emptyset = \cap_{\alpha \in \Phi^+} H_\alpha^+, \text{ where } H_\alpha^+ = \{\lambda \in V : (\lambda, \alpha) > 0\}$$

is called the fundamental chamber. More generally, for each $J \subseteq I$, put

$$_J = \begin{cases} \lambda : (\lambda, \alpha) = 0 \quad \text{for} \alpha \in J \\ \\ \lambda : (\lambda, \alpha) > 0 \text{for} \alpha \in I \setminus J \end{cases}$$

If J has only one element, then $_J$ lie on one of the hyperplanes bounding $_\emptyset$ and is called a *panel.*



Theorem 4.2 The map $w(_J) \mapsto wW_J$ is a bijection from to $\Sigma(W(\Phi), \Upsilon)$ giving an isomorphism of partially ordered sets and this put a structure of a thin chamber complex on .

([Car72] thm 15.4.1, p.288.) We call or $\Sigma(W(\Phi), \Upsilon)$ a **Coxeter complex**.

4.3 Building A **building** is a pair (Ω, A) where Ω is a chamber complex and A is a set of subcomplexes, called **apartments**, satisfying the following axioms:

B1. Ω is a thick chamber complex.

B2. Apartments in A are thin chamber complexes.

B3. Given any 2 chambers C, C' in Ω, there exists an apartment $A \in A$ such that $C \in A, C' \in A$.

B4. If C, C' are elements of Ω which are both contained in each of the apartments $A, A' \in A$ then there exists an isomorphism between A, A' leaving invariant C, C' and all their faces.

See [Tit74] §3.1. For discussions of the equivalence of definitions of buildings as chamber systems and chamber complexes see [Scha95]; [Tit74] 3.7; [Tit81] 2.2.

§5. Conclusion

The Erlangen program was published by Felix Klein in 1872 as Vergleichende Betrachtungen äber neuere geometrische Forschungen. It is named after the University Erlangen-Nürnberg, where Klein worked. One of the themes of this program says that given a geometric structure S there is a group G such that the properties of S can be characterized as invariants of G. For example if S is the projective space ^{n} then we take $G = PGL(n)$. Conversely given a group G Tits constructs a geometric structure S on which G acts as transformations preserving its geometric properties; Tits called S a building. See [JiP15] for further discussions.

A real connected semisimple Lie group G with finite center has an Iwasawa decomposition $G = KAN$ where K is a maximal compact subgroup of G. This decomposition is very useful for the study of the classical symmetric spaces represented as a quotient $K\backslash G$. If G is now the group of rational points of a connected semisimple linear algebraic group over a p-adic local field, then theory of building was used to give a proof that G also has an Iwasawa decomposition $G = KAN$ where K is a suitable choice of a maximal compact subgroup of G. The question was then asked : what is the analogue of he classical symmetric space "$K\backslash G$" in the p-adic case? The answer given by Tits was a "building".

This paper is written for students in China and so it is not for experts who know everything anyway, nor is it for students in Beijing and Shanghai where they can learn from experts visiting or in residence. This set of notes aims to be an illustration of buildings for students who live in those parts of China where they may not find a teacher in building theory or where access to printed materials on building is difficult. We assume that such a student wants a quick introduction so as to learn how to use this theory and may spend the time to read the originals later to get a firmer grasp. We have tried to start from almost no requirements on background but from the moment an algebraic group enters heavy machinery becomes unavoidable. Fortunately linear algebraic groups were once taught in some places in China. Had we found a journal to publish it in Chinese this would have been written in Chinese.

The paper has four chapters. The first chapter gives a description of buildings as an incidence geometry with a group action - this is elementary. The second chapter deals with the basic examples of buildings - to read this chapter you need to know some Lie algebra. The next chapter is based on the theory of linear algebraic groups and so this is more difficult. The final chapter is the most demanding one on your background - here we use algebraic and rigid geometry.

This is not a survey and yet it goes very fast and deep on the subjects we have selected. Let us explain. To the creator of the theory of building - Jacques Tits (Abel Prize 2008), the first thing about a building is that it is a 'geometry', in particular an incidence geometry. Now in China the moment the word geometry is mentioned people think that you are talking about differential geometry, in particular theory of partial differential equations on manifolds. But that is NOT geometry, that is properties of differential equations. 'Geometry' is about the nature of space. Just as to Einstein the theory of general relativity is about the nature of space-time, to Tits the theory of building is about the nature of space/group. So in the first chapter we try to introduce the idea of a 'geometry' through the Euclidean plane geometry. We give a short account on incidence geometry and projective space based on lectures of Tits which may not be easily available. Then we quickly jump to the axioms of a building hoping that the analogy with Euclidean plane geometry and projective geometry would lead the reader to attempt to conceive a building as a geometry - this requires some efforts. We shall return to this point later in the paper. We give two definitions of buildings - one as a chamber system following the modern expositions of Weiss [Wei03], [Wei09]; and the other as a chamber complex.

In the second chapter we present Tits discovery - how to get a 'geometry' out of a group. This construction is now called a Tits system. We quickly cover this and then we illustrate this construction with matrix groups - to be specific with the Chevalley groups. The structure theory of linear algebraic group was first introduced into China by Lai in 1979. Since then it soon was no longer taught and nowadays this theory is unknown to most algebra graduate students in China. So after a short introduction the students will face another jump - an illustration of the structure of semisimple linear algebraic group through classical Chevalley groups. We shall discuss the buildings associated to these Chevalley groups in two cases, namely over a finite field and over a local field. These corresponds to the two major type of buildings, namely the spherical buildings and the affine buildings.

Any finite extension of the field of p-adic numbers is a local field - a subject studied in books on algebraic number theory. In the third chapter we shall use the structure theory of reductive groups over local fields to present the construction of affine buildings. This is really not a theory that is an end in itself, rather it is the beginning of an important and beautiful theory - the representation theory of p-adic Lie groups. All known fundamental results in the representation theory of Lie groups are based on detail knowledge of the structure of these groups, the p-adic case is no different. The theory of Bruhat-Tits building provides us with such detail information. At the end of this chapter we will point the students to some work done in the recent years using the theory of Bruhat-Tits buildings and the theory of arithmetic differential equations (D-modules) to prove new results on the representations of p-adic Lie groups. We end this chapter with a revision note on the multifarious Hecke algebras.

The fourth chapter will be on p-adic geometry of homogeneous space constructed using buildings - extending the work first pioneered by Mumford (Fields Medal 1974, Shaw Prize 2006, Wolf Prize 2008)[Mumf72] and Drinfel'd (Fields Medal 1990, Wolf Prize 2018) [Drin76].

A building is a combinatorial object with a "metric" given by a Coxeter system. When the Coxeter group of the Coxeter system is finite we say that it is a spherical building and if the Coxeter group is infinite then it is called an affine (or an euclidean) building. The simplest building is a tree, see the book ([Serr80]) by Serre (Fields Medal 1954, Wolf Prize 2000, Abel Prize 2003).

From a semisimple linear algebraic groups G defined over a field F we construct a building $_G$. When F is a finite field $_G$ is a spherical building; when F is a local field it is an affine building. $_G$ provides important information on the structure of G, and hence they play an important role in the representation theory of G, for this reason we encourage students to learn some properties of buildings.

There are at least two pieces of mathematics in which theory of building plays a key role -
(1) Klein's Erlangen program for geometry,
(2) the combinatorial structure of reductive algebraic groups.

Langlands (Wolf Prize 1996, Shaw Prize 2007, Abel Prize 2018) placed reductive groups at the center of his theory of L-functions. To make calculations over local fields one needs to use the the combinatorial structure of reductive groups over local fields. As a first example is his calculation of orbital integrals of $GL(2)$ using building. Though the basic result in this direction - the so called Fundamental lemma - was proved by Ngô Bau Châu (Fields Medal 2010) by very different methods, recent unfinished work on representation of reductive groups over p-adic fields in p-adic topological spaces made critical use of theory of buildings. Thus revive much interests in this theory.

There are indeed many books on theory of building for example [Bro89], [Car72], [Gar97], [FN91], [Land96], [Ron89], [Serr80], [Tit74], [TW02], [Wei03], [Wei09], [Woes00]. [Ji06], [Ron92], [Scha95] are excellent survey papers. But here we deal with the more practical matters such as definitions, examples, tables for computation and so serve a different purpose.

The first difficulty about learning this theory is that the first papers [BrT67], [Tit79] contains almost no proofs. The second difficulty is that the papers of Bruhat and Tits from 1972 to 1987 are very long, they are written sometimes in very general terms which does not help the uninitiated students to understand and indeed they had some unexpected mistakes in their original ideas of proof (after 1972, the next paper was published 12 years later, but still did not complete the proofs of claims in 1979). At the moment there are a few books covering either the pure combinatorics or scratching the surface of the affine theory so a tool box summary can still be a useful guide. Even though what we have presented is public knowledge well known to all experts, but our organization and comments still have its novelty and could be useful to those students on their own. After we have finished the paper, looking back we have to admit that for a student to learn everything here without a teacher will be a difficult task that requires a lot patience and perseverance. A diligent student will have to invent many proofs!

We come to the end of the first chapter. We hope we have given you some ideas of a building as a geometry. At this moment the definitions are still rather abstract. In the next part we shall illustrate how these definitions work by constructing some examples of buildings. For more information on graphs consult [BM]. If you want more adventures in "combinatorial geometry" you can read this paper of the famous Gelfand [GS87] or look at [BGW], [Bjo]. But this is really a good time to read a book on building, we recommend the book [Wei03] of Weiss - it is short only 130 pages, you do not have to know anything and can begin now!

[References]

[1] Adams J F, Lectures on Exceptional Lie Groups, University of Chicago Press, 1996.

[2] Artin E, Elements of Algebraic Geometry, Courant Institute of Mathematical Sciences, New York University, 1st edition (1955); reprinted by Forgotten Books (2018).

[3] Artin E, Geometric algebra, Wiley Interscience, New York, 1957.

[4] Ash A, Mumford D, Rapoport M, Tai Y, Smooth Compactification of Locally Symmetric Varieties. Math. Sci. Press, Brookline, Massachusetts (1975).

[5] Baily W, Borel A, Compactification of arithmetic quotients of bounded symmetric domains, Ann. of Math., 84 (1966), 442-528.

[6] Bialyniki-Birula A, Sommese A, Complete quotients by algebraic tour actions, in Springer Lect Notes Math 956 (1982) 10-22.

[7] Björner A, Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings, Adv. in Math., 52 (1984), pp. 173212.

[8] Bondy A, Murty U, Graph Theory (Graduate Texts in Mathematics), Springer, 2008.

[9] Bornmann M, Ganzzahlige affine Hecke Algebren. Diplomarbeit, Münster, 2009.

[10] Borel A, Groupes linéaires algébriques, Annals of Math. (2) 64 (1956) 20-80.

[11] Borel A, *Linear algebraic groups*, Berlin, Springer, 1991.

[12] Borel A, Ji L, *Compactifications of symmetric and locally symmetric spaces*, Birkhauser, 2006.

[13] Borel A, Serre J-P, Corners and arithmetic groups, Comm. Math. Helv. 48 (1973) 436 -491.

[14] Borel A, Serre J-P, Cohomologie d'immeubles et de groupes S-arithmetiques, Topologie 15 (1976), 211-232.

[15] Borel A, Springer T, Rationality properties of linear algebraic groups II, Tohoku Math. J. 20 (1968), 443-497

[16] Borel A, Tits J, Groupes reductifs. Publ. Math. I.H.E.S. 27 (1965), 55-150; Complements 41 (1972), 253-276.

[17] Borovik A, Gelfand I, White N, Combinatorial flag varieties, Journal of combinatorial theory, Series A, 91 (2000), pp. 111-136.

[18] Bosch S, *Lectures on formal and rigid geometry*, New York, Springer, 2014.

[19] Bourbaki N, Groupes et algebres de Lie, chapters 4, 5 and 6. Masson, Paris. 1981.

[20] J. Boutot J, Carayol H, Uniformisation p-adique des courbes de Shimura: les thormes de Cherednik et de Drinfeld, in Courbes modulaires et courbes de Shimura, Astrisque 196-197 (1991), 7, 45158.

[21] Brown K. Buildings, Springer-Verlag, New York, 1989.

[22] Bruhat F , Tits J. Groupes algébriques simples sur un corps local. In Proceedings of a Conference on Local Fields (Driebergen, 1966, T. A. Springer, ed.), pp. 23-36. Springer. 1967.

[23] Bruhat F , Tits J. Groupes réductifs sur un corps local, I. Données radicielles valuées. Publ. Math. I.H.E.S. 41 (1972), 5-252.

[24] Bruhat F , Tits J. Groupes réductifs sur un corps local, II. Schémas en groupes. Existence dune donnée

radicielle valuée. Publ. Math. I.H.E.S. 60, (1984) 5-184.

[25] Bruhat F , Tits J. Schémas en groupes et immeubles des groupes classiques sur un corps local, I. Bull. Soc. Math. Franc. 112, 1984, 259-301.

[26] Bruhat F , Tits J. Schémas en groupes et immeubles des groupes classiques sur un corps local, II. Groupes unitaires. Bull. Soc. Math. Franc. 115,1987, 141-195.

[27] Bruhat F , Tits J. Groupes algébriques sur un corps local, III. Compléments et applications la cohomologie galoisienne. J. Fac. Sci. Univ. Tokyo IA Math. 34, 1987, 671-698.

[28] B. de Bruyn, An Introduction to Incidence Geometry, Birkhauser (2016).

[29] F. Buekenhout F. Handbook of incidence geometry, North Holland, 1995.

[30] Carayol H. Nonabelian Lubin-Tate theory, in Automorphic forms, Shimura varieties, and L-functions, Vol. II 1539, Perspect. Math., 11, Academic Press, Boston, MA, 1990.

[31] Carter R. Simple groups of Lie type, John Wiley, New York, 1972.

[32] Cartier P. Representations of p adic groups, In Proc. Symp. Pure Math. 33, Part 1 (Automorphic Forms, Representations and L- Functions, Corvallis 1977), pp. 111-156. Amer. Math. Soc., Providence, 1979.

[33] Casselman W. The unramified principal series of p-adic groups, I, Compo.Math. 40 (1980) 387-406.

[34] Casselman W, Shalika J. The unramified principal series of p-adic groups, II, Compo.Math. 41 (1980) 207-231.

[35] Chai C, Faltings G. Degeneration of abelian varieties, Springer, 1991.

[37] Cherednik I, Uniformization of Algebraic curves by Discrete Arithmetic Subgroups of PGL2 with Compact Quotients. Math.U.S.S.R. Sbornik 29 (1976), 55-78.

[38] Conrad B. et.al. Autour des schmas en groupes. Vol. I. Panoramas et Synthèses, 42/43, 2014. Vol. II. Panoramas et Synthèses, 46, 2015. Société Mathématique de France.

[39] Coxeter H. Projective Geometry, Springer, 1973.

[40] Curtis M., Matrix Groups, Springer, 1984.

[41] V. Danilov V. Geometry of toric varieties, Russian Math. Survey 33(1978)97-154.

[42] Diamond F, Shurman J. A first course course in modular forms, Springer, 2005.

[43] Drinfel'd V, Coverings of p-adic symmetric regions, Functionl anal appli. 10(1976) 107-115.

[44] Elman R, Karpenko N, Merkurjev A. The Algebraic and Geometric Theory of Quadratic Forms (Colloquium Publications), Amer. Math. Soc., 2008.

[45] Fig-Talamanca A, Nebbia C. Harmonic analysis and representation theory for groups acting on homogeneous trees, Cambridge University Press, 1991.

[46] Flaschka H, Haine L. Torus orbits on G/P, Pacific J. Math. 149 (1991) 251-292.

[47] Fu L. *Algebraic Geometry*, Tsinghua University Press, Beijing, 2006.

[48] Fulton W, Introduction to toric varieties, Princeton University press, 1993.

[49] Garrett P, Buildings and classical groups, Chapman & Hall, London, 1997.

[50] Gekeler E, Drinfeld Modular Curves, Lect. Notes in Math .1231, Springer-Verlag (1986).

[51] Gel'fand I, Serganova V. Combinatorial geometries and torus strata on homogeneous compact manifold, Russsian Math. Survey, 42(1987) 133-168.

[52] Gerritzen L, van der Put M, Schottky Groups and Mumford Curves, Lect. Notes in Math. 817 , Springer-Verlag (1980).

[53] Gérardin P, On harmonic functions on symmetric spaces and buildings, Proceedings Canadian Math. Soc. Conference 1980, American Math Soc, Providence, R.I.

[54] Gérardin P, Harmonic functions on buildings of reductive split groups. In: Operator Algebras and Groups Representations. Pitman (1984), 208-221.

[55] Gérardin P, Lai K.F. Asymtotic behaviour of eigenfunctions for the Hecke algebra on homogeneous trees, in Special Functions, ed. C Dunkl, World Scientific Pub. Co. 2000, 114-117.

[56] Gérardin P, Lai K.F. Asymtotic behaviour of eigenfunctions on semi-homogeneous trees, Pacific Journal of Mathematics 196 (2000) 415-427.

[57] Graham R, Grotschel M, Lovasz L. Handbook of Combinatorics Volumes 1 & 2, North Holland, 1995.

[58] Griffiths P, Harris J, Principles of Algebraic Geometry, Wiley-Interscience, 1978.

[59] Gross B, Parahorics, Harvard Notes (2012)

[60] Haines T, Rapoport M, On parahoric subgroups, Advances in Math 219 (2008) 188-198.

[61] Hecke E, Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I, II, Math. Ann., 114 (1937), 1-28,316-351 (=Math. Werke, 644-707).

[62] Helgason S, Differential geometry, Lie groups and symmetric spaces, Academic Press (1978)

[63] Hu Y, Geometry and topology of quotient varieties of torus actions, Duke Math J, 68 (1992) 151.

[64] Humphreys J. *Introduction to Lie algebras and representation theory*, New York, Springer, 1972.

[65] Humphreys J. *Linear algebraic groups*, New York, Springer, 1975.

[66] Humphreys J. *Reflection groups and Coxeter groups*, Cambridge University Press. 1990.

[67] Huyghe C, Schmidt T, Strauch M, Arithmetic structures for differential operators on formal schemes, preprint.

[68] Iwahori N, Matsumoto H, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, Inst. Hautes Itudes Sci. Publ. Math., 25 (1965), pp. 548.

[69] Jacquet H, Langlands R. Automorphic forms on GL(2), Lecture notes in mathematics, Springer, New York, 1970.

[70] Jacobson N, Exceptional Lie algebra, Dekker, New York, 1971.

[71] Ji Lizhen, Buildings and their applications in geometry and topology, Asian J. Math. 10 (2006), 11-8.

[72] Ji Lizhen, Papadopoulos A, Sophus Lie and Felix Klein, The Erlangen Program and its Impact in Mathematics and Physics, European Math Society Publishing House (2015).

[73] Kashiwara M, Kowata A, Minemura K, Okamoto K, Oshima T, Tanaka M. Eigenfunctions of invariant differential operators on a symmetric space, Ann. Math., 107 (1978), 1-39.

[74] Kazhdan D, Lusztig G, Proof of the DeligneLanglands conjecture for Hecke algebras, Invent. Math. 87(1) (1987), 153-215.

[75] Kempf G, Knudsen F, Mumford D, Saint-Donat B, Toroidal embeddings, 1, Lect. Notes Math. 339, Springer, 1973.

[76] Kneser M, Lectures on Galois cohomology of classical groups, Tata Institute, Bombay, 1969.

[77] Koranyi A, Harmonic functions on trees and buildings, Contemporary Mathematics, vol. 206, American Mathematical Society, 1997.

[78] Kostant B, Groups over Z, Algebraic Groups and Discontinuous Subgroups, Proc. Symp. Pure Math. IX, Providence: Amer. Math. Soc., 1966.

[79] Kurihara A, On some Examples of Equations defining Shimura Curves and the Mumford Uniformization. J . Fac . Sci. Univ. Tokyo , Sec . IA, 25 (1979), 277-300.

[80] Kurihara A, Construction of p-adic unit balls and the Hirzebruch proportionality, Amer J. Math. 102(1980) 565-648.

[81] Lai K F, Voskuil H, p-adic automorphic functions for the unitary group in three variables, Algebra Colloquium 7 (2000) 335-360.

[82] Lai K F. Orders of Finite Algebraic Groups, Pacific J. of Math., 97 (1981) 425-435.

[83] Lai K F. C_2 buildings and projective space, J. Aust. Math. Soc. 76 (2004) 383-402.

[84] Lai K F, et. al. Introduction to algebraic groups, Science Press, Beijing, China (2006) (In Chinese)

[85] K. F.Lai, Diffrential equations and Lie group representations, Advances in mathematics (China), 48 (2019) 257-301 (in Chinese).

[86] Landvogt E, A compactification of the Bruhat-Tits building, Lecture Notes in Mathematics, vol. 1619, Springer-Verlag, Berlin, 1996.

[87] Langlands R, Base change for GL(2), Annals of Math Studies, Princeton University Press (1980).

[88] Lusztig G, Affine Hecke algebras and their graded version, JAMS 2(3) (1989), 599-635.

[89] Macdonald I, Spherical functions on a p-adic group, University of Madras Press, 1971.

[90] Milne J, Algebraic groupes, Cambridge University Press, 2017.

[91] Mumford D. Geometric invariant theory, Springer (1965).

[92] Mumford D. An analytic construction of degenerating curves over complete local rings, Compositio Math. 24 (1972) 129-174.

[93] Mumford D. Algebraic Geometry I: Complex Projective Varieties, Springer, Corr. 2nd printing 1981.

[94] Mustafin G, Nonarchimedean uniformization, Kath. USSR Sbornik, 34 (1978) 187-214.

[95] Newstead P, Moduli problems and orbit spaces, TATA, Bombay, 1978.

[96] Oda T, Torus embedding, TATA, Bombay, 1978.

[97] Ollivier R, Schneider P, Pro-p Iwahori-Hecke algebras are Gorenstein, J. Inst. Math. Jussieu 13 (2014), 753-809.

[98] Parkinson J, Spherical harmonic analysis on affine buildings. Mathematische Zeitschrift, 253(3) (2006), 571-606.

[99] Patel D, Schmidt T, Strauch M, Locally analytic representations and sheaves on the Bruhat-Tits building, Algebra and Number Theory, 2014, 8 (6) , 1365 - 1445.

[100] Piatetski-Shapiro I, Automorphic functions and geometry of classical domains, Gordon and Breach, New York, 1969.

[101] Porteous I, Topological geometry, Van Nostrand Reinhold, New York, 1969.

[102] van der Put M, Voskuil H, Symmetric space associated to split algebraic groups over a local field, J. Reine Angew. Math. 433(1992) 69-100.

[103] Ronan M, Buildings: main ideas and applications. I. & II, Bull. London Math. Soc., 24 (1992), 1-51; 24 (1992), 97-126.

[104] Ronan M, *Lectures on buildings.* Perspectives in Mathematics, vol. 7, Academic Press, 1989.

[105] Ronan M, Smith S, Sheaves on buildings and modular representations of Chevalley groups, J. Algebra 96 (1985), 319-346.

[106] Satake I, On representations and compactifications of symmetric Riemannian spaces, Ann. of Math. , 71 (1960) 77-110.

[107] Satake I, Classification theory of semi-simple groups, Marcel Dekker, !971.

[108] Satake I, Algebraic structures of symmetric domains, Princeton University Press, 1980.

[109] Scharlau R, Buildings, in Handbook of incidence geometry, 477-645, North-Holland, 1995.

[110] Schneider P, Stuhler U, Representation theory and sheaves on the Bruhat-Tits building, Inst. Hautes Etudes Sci. Publ. Math., 1997, (85), 97191.

[111] Seidenberg A, Lectures in projective geometry,Dover, 2006.

[112] Serre J, Trees, Springer-Verlag, 1980.

[113] Shimura G, Sur les integrales attachees aux formes automorphes, J. Math. Soc. Japan, 11 (1959), 291-311.

[114] Shimura G, Introduction to the arithmetic theory of automorphic funtions, Princeton University Press, 1971.

[115] Springer T. Some arithmetical results on semi-simple Lie algebras, Publ. math, IHES, 30 (1960) 115-141.

[116] Springer T. *Linear algebraic groups*, Boston, Birkhauser, 1998.

[117] Springer T, Veldkamp D, *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monographs in Mathematics, 2010.

[118] Steinberg R, Variations on a Theme of Chevalley. Pac. J. of Math. IX (1959), 875-891.

[119] Steinberg R, Regular elements of semi-simple algebraic groups. Publ. Math. (IHES) 25 (1965), 49-80.

[120] Steinberg R, Lectures on Chevalley groups, mimeographed lecture notes, New Haven, Conn.: Yale Univ. Math. Dept. 1968.

[121] Sugiura M, Conjugate classes of Cartan subalgebras in real semi-simple Lie algebras, J. Math. Soc. Japan, 11(1959)374-434, 23(1971)314-383.

[122] Tamagawa T, On the zeta functions of a division algebra, Ann. of Math., 77 (1963), 387-405.

[123] Taylor D, The geometry of classical groups, Heldermann Verlag, 1992 (ISBN : 3-88538-009-9).

[124] Tits J. Groupes algebriques seni-simples et geometries associees, in, Proc. Coll. Algebraical and Topological Foundations of Geometry, at Utrecht 1959, Pergamon Press, 1962, 175-192.

[125] Tits J. Algebraic and abstract simple groups, Annals Math. 80 (1964) 313-329.

[126] Tits J. Classification of algebraic semi-simple groups. In Algebraic Groups and Discontinuous Groups (Boulder, 1965), pp. 33-62, Proc. Symp. Pure Math. 9. Amer. Math. Soc., Providence, 1966.

[127] Tits J. Buildings of Spherical Type and Finite BN-Pairs, Lecture Notes in Mathematics, vol. 386. Springer. 1974.

[128] Tits J. Reductive groups over local fields. In Proc. Symp. Pure Math. 33, Part 1 (Automorphic Forms, Representations and L- Functions, Corvallis 1977), pp. 29-69. Amer. Math. Soc., Providence, 1979.

[129] Tits J. A local approach to buildings, In *The Geometric Vein* (Coxeter festschrift) (eds. C. Davis, B, Grunbaum, F. Sherk) pp. 519-547, Springer, 1981.

[130] Tits J. Lectures on buildings and arithmetic groups, Yale University Math. Dept., 1984.

[131] Tits J, Weiss R, Moufang Polygons. Springer.2002.

[132] Ueberberg J, Foundations of incidence geometry, Springer, 2011.

[133] Vigneras M, Representations ℓ-modulaires dun groupe reductif p-adique avec $\ell \neq p$, Progress Math. 137 (1996), Birkhauser.

[134] Vigneras M, Pro-p-Iwahori Hecke ring and supersingular$\bar{}_p$ -representations, Math. Ann. 331 (2005), 523-556 (erratum: vol. 333(3), p. 699-701).

[135] Vigneras M, The pro-p-Iwahori Hecke algebra of a reductive p-adic group, I. Compositio mathematica 152, vol.7 No1, 2016, 653-753; II. Muenster J. of Math. Vol. 7, No 1, 2014 (364-379); III. Journal of the Institute of Mathematics of Jussieu 2015, 1-38; V. Pacific J. of Math. vol. 279 No 1-2, 2015, 499-529.

[136] Weil A. Basic number theory, Springer, 1974.

[137] Weiss R. The Structure of Spherical Buildings. Princeton University Press. 2003.

[138] Weiss R. The structure of affine buildings. Princeton University Press. 2009.

[139] Woess W. Random walks on infinite graphs and groups, in Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, 2000.

[140] Wolf J. Fine structures of Hermitian symmetric space, 271-351 in "Symmetric spaces" Dekker, New York, 1972.

[141] Xu Yichao. Theory of complex homogeneous bounded domains, Mathematics and its Applications, 569, Beijing: Science Press (2005).