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The Maschke-type Theorems of Yetter-Drinfeld Hopf Algebras

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Abstract: In this paper, we give the Maschke-type theorem for a Yetter-Drinfeld Hopf algebra which extends the famous results for a usual Hopf algebra[3].

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It is well-known that a Hopf algebra with a projection has the Radford biproduct decomposition [4]. In this decomposition one of the factors is not a Hopf algebra in the usual sense, but it is a Hopf algebra in the category of Yetter-Drinfeld modules over the other factor(the projection). There are other important examples of Hopf algebras in braided category such as (G,χ)-Hopf algebras and twisted Hopf algebras ([2]). With the deeper researches many interesting results are obtained. For example, Doi considered Hopf modules in Yetter-Drinfeld category ([5]) and Schauenburg proved that the category of Yetter-Drinfeld modules is equivalent to the category of modules over Drinfeld double ([6]).

In this paper we mainly give the Maschke-type theorem for a Yetter-Drinfeld Hopf algebra.

Conventions We work over a field k. All algebras and coalgebras are over k. For a coalgebra H we denote $\Delta(h)=\sum h_1\otimes h_2$ and for a left H-comodule M we denote $\rho(m)=\sum m_{(-1)}\otimes m_{(0)}$. id_V denotes the identity map from V to itself for any k-space V.

We first recall some basic notions. Suppose that $(H,m,\mu,\Delta,\varepsilon,S)$ is a Hopf algebra. A left Yetter-Drinfeld modules over H is a k-space M is endowed with both a left H-module structure by . and a left H-comodule structure by ρ verifying $\sum (h\cdot m)_{(-1)}\otimes (h\cdot m)_{(0)}=\sum h_1 m_{(-1)}S(h_3)\otimes h_2\cdot m_{(0)}$. The category of Yetter-Drinfeld modules over H denoted by ${}^H_H YD$

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is a prebraided category with the prebraiding given by $\tau_{M,N}: M\otimes N\to N\otimes M,\ \tau_{M,N}(m\otimes n)=\sum m_{(-1)}\cdot n\otimes m_{(0)}$ for two Yetter-Drinfeld modules M and N.

Definition 1[5] Let H be a Hopf algebra. An object L in ${}_H^HYD$ is called a Yetter-Drinfeld bialgebra if it is a k-algebra and also a k-coalgebra satisfying the following conditions for all $h\in H,\ l,j\in L$:

(1) L is an H-module algebra via the action $\cdot$, that is,

$$h\cdot(lj)=\sum(h_1\cdot l)(h_2\cdot j),\ h\cdot 1_L=\varepsilon(h)1_L;$$

(2) L is an H-module coalgebra via the action $\cdot$, that is,

$$\Delta(h\cdot l)=\sum h_1\cdot l_1\otimes h_2\cdot l_2,\ \varepsilon(h\cdot l)=\varepsilon(h)\varepsilon(l);$$

(3) L is an H-comodule algebra via the coaction ρ, that is,

$$\rho(lj)=\sum(lj)_{(-1)}\otimes(lj)_{(0)}=\sum l_{(-1)}j_{(-1)}\otimes l_{(0)}j_{(0)},\ \rho(1_L)=1_H\otimes 1_L;$$

(4) L is an H-comodule coalgebra via the coaction ρ, that is,

$$\sum l_{(-1)}\otimes l_{(0)1}\otimes l_{(0)2}=\sum l_{1(-1)}l_{2(-1)}\otimes l_{1(0)}\otimes l_{2(0)},\ \sum l_{(-1)}\varepsilon(l_{(0)})=\varepsilon(l)1_H;$$

(5) Δ and ε are algebra homomorphisms in ${}_H^HYD$, that is,

$$\Delta(lj)=\sum l_1(l_{2(-1)}\cdot j_1)\otimes l_{2(0)}j_2,\ \Delta(1)=1\otimes 1,\ \varepsilon(1)=1,\ \varepsilon(lj)=\varepsilon(l)\varepsilon(j).$$

In addition, if there exists a morphism $S: L\to L$ in ${}_H^HYD$ such that it is the convolution inverse to id_L, then L is called a Yetter-Drinfeld Hopf algebra over H.

Lemma 1[5] Let L be a Yetter-Drinfeld Hopf algebra over H with the antipode S, then for any $l,l'\in L,\ S(ll')=\sum(l_{(-1)}\cdot S(l'))S(l_{(0)}),\ S(1)=1,\ \Delta(S(l))=\sum l_{1(-1)}\cdot S(l_2)\otimes S(l_{1(0)}),\ \varepsilon(S(l))=\varepsilon(l)$.

Definition 2 Let L be a Yetter-Drinfeld Hopf algebra over H, an element $t\in L$ is called a right integral if $tl=\varepsilon(l)t$ for any $l\in L$. In addition, if $\varepsilon(t)=1$ and $\sum t_{(-1)}\otimes t_{(0)}=1\otimes t$ then it is called a coinvariant normalized right integral.

Example 1 $L=k^{Z_2}\otimes k[Z_2]$ is a Yetter-Drinfeld Hopf algebra over $H=k[Z_2]$ under the following structures form [7]:

1. The multiplication: $(e_i\otimes c_u)(e_j\otimes c_v)=\delta_{ij}\sigma_i(u,v)e_i\otimes c_{u+v}$; the unit $1=\sum_{i=0}^1 e_i\otimes c_0$.

2. The comultiplication: $\Delta(e_i\otimes c_u)=\sum_{s=0}^1(e_s\otimes c_u)\otimes(e_{i-s}\otimes c_u)$; the counit $\varepsilon(e_i\otimes c_u)=\delta_{i0}$.

3. The H-action: $c_u\cdot(e_i\otimes c_v)=(-1)^{ivu}e_i\otimes c_v$.

4. The H-coaction: $\sum(e_i\otimes c_v)_{(-1)}\otimes(e_i\otimes c_v)_{(0)}=\sum c_{iv}\otimes(e_i\otimes c_v)$.

5. The antipode: $S(e_i\otimes c_u)=\sigma_i(u,-u)^{-1}e_{-i}\otimes c_{-u},\ \sigma_0(i,u)=1,\ \sigma_1(i,u)=\iota^{q(i,u)},\ q(i,u)=1$ if and only if $i=u=1$, otherwise $q(i,u)=0,\ \iota$ is a primitive fourth root of unity.

Then we claim that $t=\frac{1}{2}(e_0\otimes c_0+e_0\otimes c_1)$ is a coinvariant normalized right integral. In fact,

$$\begin{aligned}
&t(e_0\otimes c_0)=\tfrac{1}{2}(e_0\otimes c_0+e_0\otimes c_1)=\varepsilon(e_0\otimes c_0)t;\\
&t(e_0\otimes c_1)=\tfrac{1}{2}(e_0\otimes c_1+e_0\otimes c_0)=\varepsilon(e_0\otimes c_1)t;\\
&t(e_1\otimes c_0)=0=\varepsilon(e_1\otimes c_0)t;\quad t(e_1\otimes c_1)=0=\varepsilon(e_1\otimes c_1)t;\\
&\rho(t)=\tfrac{1}{2}c_0\otimes(e_0\otimes c_0)+\tfrac{1}{2}c_0\otimes(e_0\otimes c_1)=c_0\otimes t;\\
&\varepsilon(t)=\tfrac{1}{2}(\varepsilon(e_0\otimes c_0)+\varepsilon(e_0\otimes c_1))=1.
\end{aligned}$$

Definition 3 Let L be a Yetter-Drinfeld Hopf algebra over a Hopf algebra H (via the action $\cdot$ and the coaction ρ_L) and A an algebra in ${}^H_HYD$ (via the action $\rightharpoonup$ and the coaction ρ_A). A is called *a left Yetter-Drinfeld L-module algebra* if A is a left L-module (via the action $\circ$) satisfying the following for all $h\in H,\ y\in L,\ a,b\in A$:

(6) $y\circ(ab)=\sum(y_1\circ(y_{2(-1)}\rightharpoonup a))(y_{2(0)}\circ b),\ y\circ 1_A=\varepsilon(y)1_A$;

(7) $\sum(h_1\cdot y)\circ(h_2\rightharpoonup a)=h\rightharpoonup(y\circ a)$;

(8) $\sum(y\circ a)_{(-1)}\otimes(y\circ a)_{(0)}=\sum y_{(-1)}a_{(-1)}\otimes y_{(0)}\circ a_{(0)}$.

Example[8] Let L be a Yetter-Drinfeld Hopf algebra over a Hopf algebra H (via the action $\cdot$ and the coaction ρ). Note that L is a left Yetter-Drinfeld L-module algebra with the action $\circ$ given by $y\circ z=\sum y_1(y_{2(-1)}\cdot z)S(y_{2(0)}),\ y,z\in L$. Indeed, obviously $1\circ x=x$ for any $x\in L$. We have to show the following equations hold. In fact, for any $x,y,z\in L$,

$$\begin{aligned}
&x\circ(y\circ z)=\sum x_1[x_{2(-1)}\cdot(y_1(y_{2(-1)}\cdot z)S(y_{2(0)}))]S(x_{2(0)})\\
&\overset{(1)}{=}\sum x_1(x_{2(-1)1}\cdot y_1)(x_{2(-1)2}y_{2(-1)}\cdot z)(x_{2(-1)3}\cdot S(y_{2(0)}))S(x_{2(0)})\\
&\overset{Lemma1}{=}\sum x_1(x_{2(-1)}\cdot y_1)(x_{2(0)(-1)}y_{2(-1)}\cdot z)S(x_{2(0)(0)}y_{2(0)})\\
&\overset{(5)}{=}\sum (xy)_1((xy)_{2(-1)}\cdot z)S((xy)_{2(0)})=(xy)\circ z;
\end{aligned}$$

$$\begin{aligned}
&\sum(x_1\circ(x_{2(-1)}\cdot y))(x_{2(0)}\circ z)\\
&=\sum x_1(x_{2(-1)}x_{3(-1)}\cdot y)S(x_{2(0)})x_{3(0)1}(x_{3(0)2(-1)}\cdot z)S(x_{3(0)2(0)})\\
&\overset{(4)}{=}\sum x_1(x_{2(-1)}\cdot y)S(x_{2(0)1})x_{2(0)2}(x_{2(0)3(-1)}\cdot z)S(x_{2(0)3(0)})\\
&=\sum x_1(x_{2(-1)}\cdot y)(x_{2(0)(-1)}\cdot z)S(x_{2(0)(0)})\\
&=\sum x_1(x_{2(-1)1}\cdot y)(x_{2(-1)2}\cdot z)S(x_{2(0)})=x\circ(yz);
\end{aligned}$$

$$\begin{aligned}
&\sum(h_1\cdot x)\circ(h_2\cdot y)=\sum(h_1\cdot x_1)((h_2\cdot x_2)_{(-1)}h_3\cdot y)S((h_2\cdot x_2)_{(0)})\\
&=\sum(h_1\cdot x_1)(h_2x_{2(-1)}S(h_4)h_5\cdot y)S(h_3\cdot x_{2(0)})\\
&=\sum(h_1\cdot x_1)(h_2x_{2(-1)}\cdot y)S(h_3\cdot x_{2(0)})\\
&=\sum h\cdot(x_1(x_{2(-1)}\cdot y)S(x_{2(0)}))=h\cdot(x\circ y);
\end{aligned}$$

$$\begin{aligned}
&\sum (x\circ y)_{(-1)}\otimes(x\circ y)_{(0)}\\
&=\sum x_{1(-1)}(x_{2(-1)}\cdot y)_{(-1)}x_{2(0)(-1)}\otimes x_{1(0)}(x_{2(-1)}\cdot y)_{(0)}S(x_{2(0)(0)})\\
&=\sum x_{1(-1)}x_{2(-1)1}y_{(-1)}S(x_{2(-1)3})x_{2(-1)4}\otimes x_{1(0)}(x_{2(-1)2}\cdot y_{(0)})S(x_{2(0)})\\
&=\sum x_{1(-1)}x_{2(-1)1}y_{(-1)}\otimes x_{1(0)}(x_{2(-1)2}\cdot y_{(0)})S(x_{2(0)})\\
&\overset{(4)}{=}\sum x_{(-1)}y_{(-1)}\otimes x_{(0)1}(x_{(0)2(-1)}\cdot y_{(0)})S(x_{(0)2(0)})=\sum x_{(-1)}y_{(-1)}\otimes x_{(0)}\circ y_{(0)}.
\end{aligned}$$

From [6] and [8-9] we know that if L is a Yetter-Drinfeld Hopf algebra over H (via the action $\cdot$ and the coaction ρ) and A a left Yetter-Drinfeld L-module algebra (via the L-action $\circ$ and H-action $\rightharpoonup$), then $A\sharp L$ is an associative algebra which equals to $A\otimes L$ as a k-space with the multiplication given by $(a\sharp l)(b\sharp l^{'})=\sum a(l_1\circ(l_{2(-1)}\rightharpoonup b))\sharp l_{2(0)}l^{'}$ for all $a,b\in A,\ l,l^{'}\in L$ and the unit $1_A\sharp 1_L$.

Directly from easy computations we can obtain the following lemma.

Lemma 2 Let L be a Yetter-Drinfeld Hopf algebra over H and A be a left Yetter-Drinfeld L-module algebra , then A and L are both subalgebras of $A\sharp L$.

Lemma 3 Let L be a Yetter-Drinfeld Hopf algebra over H and A be a left Yetter-Drinfeld L-module algebra , then M is a left $A\sharp L$-module if and only if M is both a left A-module and a left L-module satisfying $\sum(l_1\circ(l_{2(-1)}\rightharpoonup a))*(l_{2(0)}\diamond m)=l\diamond(a*m)$ for any $a\in A,\ l\in L,\ m\in M$, where M is a left A-module via $*$ and a left L- module via $\diamond$.

Proof If M is a left $A\sharp L$-module via " $\cdot$", then M is a left A-module via the action given by $a*m=(a\sharp 1)\cdot m$ and a left L-module via the action given by $l\diamond m=(1\sharp l)\cdot m$ by Lemma 2. Moreover for any $a\in A,\ l\in L,\ m\in M,\ l\diamond(a*m)=[(1\sharp l)(a\sharp 1)]\cdot m=\sum[l_1\circ(l_{2(-1)}\rightharpoonup a)\sharp l_{2(0)}]\cdot m=\sum[(l_1\circ(l_{2(-1)}\rightharpoonup a)\sharp 1)(1\sharp l_{2(0)})]\cdot m=\sum(l_1\circ(l_{2(-1)}\rightharpoonup a))*(l_{2(0)}\diamond m)$.

If M is both a left A-module and a left L-module, then we claim that M is a left $A\sharp L$-module via the action defined by $(a\sharp l)\cdot m=a*(l\diamond m)$. Indeed, for all $a,b\in A,\ l,j\in L,\ m\in M$,

$$\begin{aligned}&(a\sharp l)\cdot[(b\sharp j)\cdot m]=a*[l\diamond(b*(j\diamond m))]\\&=\sum a*[(l_1\circ(l_{2(-1)}\rightharpoonup b))*(l_{2(0)}\diamond(j\diamond m))]\\&=\sum a(l_1\circ(l_{2(-1)}\rightharpoonup b))*((l_{2(0)}j)\diamond m))=[(a\sharp l)(b\sharp j)]\cdot m].\end{aligned}$$

Lemma 4 Let L be a Yetter-Drinfeld Hopf algebra over H with bijective antipode and A be a left Yetter-Drinfeld L-module algebra , and M,N be two $A\sharp L$-modules where M and N are the left L-modules via $\diamond$ and $\triangleright$ and the left A-modules via $*$ and $\star$ respectively. Let t be a coinvariant normalized integral in L. If $f:M\to N$ is a left A-module map, then the map $\overline{f}:M\to N,\ m\mapsto\sum S(t_1)\triangleright f(t_2\diamond m)$ is a left $A\sharp L$-module map.

Proof Since t is a right integral in L, we have for all $l\in L$, $\sum t_1\otimes t_2\otimes l=\Delta(t)\otimes l=\sum\Delta(t\varepsilon(l_1))\otimes l_2=\sum\Delta(tl_1)\otimes l_2=\sum t_1(t_{2(-1)}\cdot l_1)\otimes t_{2(0)}l_2\otimes l_3$. It follows that $\sum t_1\otimes t_2S(l)=\sum t_1(t_{2(-1)}\cdot l)\otimes t_{2(0)}$.

Next we will prove that $\overline{f}$ is both a left A-module map and a left L-module map. As a matter of fact, for all $l^{'}=S(l)\in L,\ m\in M$,

$$\begin{aligned}&\overline{f}(S(l)\diamond m)=\sum S(t_1)\triangleright f(t_2S(l)\diamond m)\overset{(9)}{=}\sum S(t_1(t_{2(-1)}\cdot l))\triangleright f(t_{2(0)}\diamond m)\\&\overset{Lemma2}{=}\sum(t_{1(-1)}t_{2(-1)}\cdot S(l))S(t_{1(0)})\triangleright f(t_{2(0)}\diamond m)\end{aligned}$$

$$\begin{aligned}&\overset{(4)}{=}\sum(t_{(-1)}\cdot S(l))S(t_{(0)1})\triangleright f(t_{(0)2}\diamond m)\\&=\sum S(l)\triangleright(S(t_1)\triangleright f(t_2\diamond m))=S(l)\triangleright\overline{f}(m).\end{aligned}$$

Hence $\overline{f}$ is a left L-module map. On the other hand, for all $a \in A,\ m \in M$,

$$
\begin{aligned}
&\overline{f}(a * m) = \sum S(t_1) \triangleright f(t_2 \diamond (a * m)) \\
&\stackrel{Lemma3}{=} \sum S(t_1) \triangleright f((t_2 \circ (t_{3(-1)} \rightharpoonup a)) * (t_{3(0)} \diamond m)) \\
&= \sum S(t_1) \triangleright ((t_2 \circ (t_{3(-1)} \rightharpoonup a) \star f(t_{3(0)} \diamond m))) \\
&\stackrel{Lemma3}{=} \sum [S(t_1)_1 \circ (S(t_1)_{2(-1)} \rightharpoonup (t_2 \circ (t_{3(-1)} \rightharpoonup a)))] * (S(t_1)_{2(0)} \diamond f(t_{3(0)} \diamond m)) \\
&\stackrel{Lemma1}{=} \sum [(t_{1(-1)} \cdot S(t_2)) \circ (S(t_{1(0)})_{(-1)} \rightharpoonup (t_3 \circ (t_{4(-1)} \rightharpoonup a)))] \\
&\star (S(t_{1(0)})_{(0)} \triangleright f(t_{4(0)} \diamond m)) \\
&= \sum [(t_{1(-1)1} \cdot S(t_2)) \circ (t_{1(-1)2} \rightharpoonup (t_3 \circ (t_{4(-1)} \rightharpoonup a)))] \star (S(t_{1(0)}) \triangleright f(t_{4(0)} \diamond m)) \\
&\stackrel{(7)}{=} \sum [((t_{1(-1)1} \cdot S(t_2))(t_{1(-1)2} \cdot t_3)) \circ (t_{1(-1)3} t_{4(-1)} \rightharpoonup a)] \star (S(t_{1(0)}) \triangleright f(t_{4(0)} \diamond m)) \\
&\stackrel{(1)}{=} \sum [(t_{1(-1)1} \cdot (S(t_2) t_3)) \circ (t_{1(-1)2} t_{4(-1)} \rightharpoonup a)] \star (S(t_{1(0)}) \triangleright f(t_{4(0)} \diamond m)) \\
&= \sum (t_{1(-1)} t_{2(-1)} \rightharpoonup a) \star (S(t_{1(0)}) \triangleright f(t_{2(0)} \diamond m)) \\
&\stackrel{(4)}{=} \sum (t_{(-1)} \rightharpoonup a) \star (S(t_{(0)1}) \triangleright f(t_{(0)2} \diamond m)) = \sum a \star (S(t_1) \triangleright f(t_2 \diamond m)) \\
&= a \star \overline{f}(m).
\end{aligned}
$$

So $\overline{f}$ is also a left A-module map. By the above discussion we know that $\overline{f}$ is a left $A\sharp L$-module map.

By the above Lemma, we can obtain the Maschke-type theorem for a Yetter-Drinfeld Hopf algebra.

Theorem 1 Let L be a Yetter-Drinfeld Hopf algebra over H with bijective antipode and A be a left Yetter-Drinfeld L-module algebra , and M be an $A\sharp L$-module. Let t be a coinvariant normalized integral in L and N an $A\sharp L$-submodule of M. If N is an A-direct summand of M, then N is an $A\sharp L$-direct summand of M.

Proof Let $f : M \to N$ be an A-module projection map. Define $\overline{f} : M \to N$, $m \mapsto \sum S(t_1) \diamond f(t_2 \diamond m)$, then by Lemma 4 $\overline{f}$ is an $A\sharp L$-module map.

In what follows we need only to show that $\overline{f}$ is a projection. Indeed, for all $n \in N$, $\overline{f}(n) = \sum S(t_1) \triangleright f(t_2 \diamond n) = \sum S(t_1) \triangleright (t_2 \triangleright n) = \sum S(t_1) t_2 \triangleright n = n$.

Remark 1 If we let $L := H$ and define the left H-action $\cdot$ on L as $h \cdot l = \varepsilon(h) l$ and the left H-coaction on H as $\sum l_{(-1)} \otimes l_{(0)} = 1 \otimes l$ for any $h \in H$, $l \in L$, then it is very easy to prove that H is a Yetter-Drinfeld Hopf algebra over H. Therefore the classical Maschke-type theorem for a Hopf algebra can be seen special case of this theorem.

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