

# Strongly $\alpha$ -Reflexive Rings Relative to a Monoid

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**Abstract:** For a monoid  $M$  and an endomorphism  $\alpha$  of a ring  $R$ , we introduce the notion of strongly  $M$ - $\alpha$ -reflexive rings and study its properties. For an u.p.-monoid  $M$  and a right Ore ring  $R$  with its classical right quotient ring  $Q$ , we prove that  $R$  is strongly  $M$ - $\alpha$ -reflexive if and only if  $Q$  is strongly  $M$ - $\alpha$ -reflexive, where  $R$  is  $\alpha$ -rigid,  $\alpha$  is an epimorphism of  $R$ . The relationship between some special subrings of upper triangular matrix rings and strongly  $M$ - $\alpha$ -reflexive rings is also investigated. Several known results similar to strongly  $M$ - $\alpha$ -reversible rings are obtained.

**Key words:** Unique product monoid;  $\alpha$ -reflexive ring; strongly  $M$ - $\alpha$ -reflexive ring; strictly totally ordered monoid

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## §1. Introduction

Throughout this paper,  $R$  denotes an associative ring with identity,  $\alpha$  denotes a non-zero and non-identity endomorphism, unless specified otherwise. For a ring  $R$ , the polynomial ring with an indeterminate  $x$  over  $R$  is denoted by  $R[x]$  and the  $n$  by  $n$  full matrix ring over  $R$  is denoted by  $M_n(R)$ . Also, for a monoid  $M$ ,  $e$  stands for the identity of  $M$ ,  $R[M]$  denotes the monoid ring over  $R$ . For a ring  $R$  with an endomorphism  $\alpha$ , an endomorphism of  $M_n(R)$  induced by  $(a_{ij}) \rightarrow (\alpha(a_{ij}))$  is denoted by  $\bar{\alpha}$ .

In 1981, Mason [1] first studied the reflexive property of rings. A right ideal  $I$  is reflexive if  $xRy \subseteq I$  implies  $yRx \subseteq I$  for  $x, y \in R$ .  $R$  is called reflexive if  $(0)$  is a reflexive ideal (i.e.,  $aRb = (0)$  implies  $bRa = (0)$  for  $a, b \in R$ ). Later, the properties of reflexive rings and related

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concepts were studied in [2]. According to Hashemi and Moussavi [3], an endomorphism  $\alpha$  of a ring  $R$  is called rigid if  $a\alpha(a) = 0$  implies  $a = 0$  for all  $a \in R$ . A ring  $R$  is  $\alpha$ -rigid if there exists a rigid endomorphism  $\alpha$  of  $R$ . Also, a ring  $R$  is  $\alpha$ -compatible if for each  $a, b \in R$ ,  $ab = 0$  if and only if  $a\alpha(b) = 0$ . By [3, Lemma 2.2], a ring  $R$  is  $\alpha$ -rigid if and only if  $R$  is  $\alpha$ -compatible and reduced (i.e., rings with no nonzero nilpotent elements). A ring  $R$  is called semicommutative if whenever  $ab = 0$  implies  $aRb = 0$ . Reduced rings are semicommutative, but the converse is not true. As mentioned in [4], Zhao and Zhu studied a generalization of a reflexive ring and that of an  $\alpha$ -rigid ring, which is called an  $\alpha$ -reflexive ring, whenever  $aRb = 0$  implies  $bR\alpha(a) = 0$  with a reflexive endomorphism  $\alpha$  of  $R$ . Also in [5], Kwak, Lee and Yun called it a right  $\alpha$ -skew reflexive ring. In 1997, Rege and Chhawchharia [6] introduced the notion of an Armendariz ring. They defined a ring  $R$  to be an Armendariz ring if whenever polynomials  $f(x) = a_0 + a_1x + \cdots + a_mx^m$ ,  $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$  satisfy  $f(x)g(x) = 0$ , then  $a_ib_j = 0$  for each  $i, j$ . Liu [7] called a ring  $R$  an  $M$ -Armendariz ring (an Armendariz ring relative to a monoid  $M$ ), if whenever  $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ ,  $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R[M]$  satisfy  $\alpha\beta = 0$ , then  $a_ib_j = 0$ , for all  $i, j$ . In 2014, Elshokry, Ali and Liu [8] introduced the notion of 3- $M$ -Armendariz rings, which is a generalization of an  $M$ -Armendariz ring, where  $M$  is a monoid. A ring  $R$  is called 3- $M$ -Armendariz if whenever elements  $\alpha = a_1g_1 + \cdots + a_ng_n$ ,  $\beta = b_1h_1 + \cdots + b_mh_m$ ,  $\gamma = c_1l_1 + \cdots + c_rl_r \in R[M]$ , satisfy  $\alpha\beta\gamma = 0$ , then  $a_ib_jc_k = 0$  for each  $i, j, k$ . As mentioned in [9], Singh, Juyal and Khan studied a generalization of a strongly reversible ring, which is called strongly  $M$ -reversible, whenever  $\alpha\beta = 0$  implies  $\beta\alpha = 0$  with  $\alpha, \beta \in R[M]$ . Later, Ali, Elshokry and Liu [10] presented the concept of a strongly  $M$ - $\alpha$ -reversible ring. For an endomorphism  $\alpha$  of a ring  $R$ ,  $\alpha$  is called strongly right (respectively, left) reversible relative to  $M$  if whenever  $\varphi, \psi \in R[M]$  with  $\varphi\psi = 0$ , then  $\psi\alpha(\varphi) = 0$  (resp.,  $\alpha(\psi)\varphi = 0$ ). A ring  $R$  is called strongly right (resp., left)  $M$ - $\alpha$ -reversible if there exists a strongly right (respectively, left) reversible endomorphism  $\alpha$  of  $R$  relative to  $M$ . A ring is strongly  $M$ - $\alpha$ -reversible if it is both strongly right and left  $M$ - $\alpha$ -reversible. In [11], a ring  $R$  is called strongly  $M$ -reflexive whenever  $\alpha, \beta \in R[M]$  with  $\alpha R[M]\beta = 0$ , then  $\beta R[M]\alpha = 0$ . The extensions of reflexive rings, strongly reflexive rings, Armendariz rings and reversible rings have been discussed in a number of publications (see, for example, [7], [11], [12]).

Motivated by the above facts, we introduce the concept of a strongly  $M$ - $\alpha$ -reflexive ring for an endomorphism  $\alpha$  of a ring  $R$  and a monoid  $M$ , as a generalization of an  $\alpha$ -reflexive ring and that of a strongly  $M$ -reflexive ring. We study basic examples and properties of strongly  $M$ - $\alpha$ -reflexive rings. For a monoid  $M$  and a right Ore ring  $R$  with  $Q$  its classical right quotient ring, we prove that if  $R$  is  $\alpha$ -rigid with epimorphism  $\alpha$ , then  $R$  is strongly  $M$ - $\alpha$ -reflexive if and only if  $Q$  is strongly  $M$ - $\alpha$ -reflexive. It is shown that if  $R$  is a reduced  $\alpha$ -reflexive ring,  $M$  is a u.p.-monoid, then  $T(R, R)$  is strongly  $M$ - $\bar{\alpha}$ -reflexive. At last, we prove that some special subrings of upper triangular matrix rings over a reduced ring are strongly  $M$ - $\alpha$ -reflexive under some additional conditions, where  $M$  is a strictly totally ordered monoid.

## §2. Strongly $M$ - $\alpha$ -reflexive Rings

In this section, we introduce the notion of a strongly  $M$ - $\alpha$ -reflexive ring and consider its properties and basic extensions. Some examples related to strongly  $M$ - $\alpha$ -reflexive rings are given.

Let  $\alpha$  be an endomorphism of a ring  $R$ . For  $\varphi = a_1g_1 + \cdots + a_ng_n \in R[M]$ , we denote  $\alpha(a_1)g_1 + \cdots + \alpha(a_n)g_n$  as  $\alpha(\varphi)$ . We start with the following definition.

**Definition 2.1** An endomorphism  $\alpha$  of a ring  $R$  is called strongly reflexive relative to  $M$  if whenever  $\varphi, \psi \in R[M]$  with  $\varphi R[M]\psi = 0$ , then  $\psi R[M]\alpha(\varphi) = 0$ . A ring  $R$  is called strongly  $M$ - $\alpha$ -reflexive if there exists a strongly reflexive endomorphism  $\alpha$  of  $R$  relative to  $M$ .

**Remark** (i) A ring  $R$  is strongly  $M$ -reflexive if  $R$  is strongly  $M$ - $1_R$ -reflexive, where  $1_R$  is the identity endomorphism of  $R$ . (ii) Every subring  $S$  with  $\alpha(S) \subseteq S$  of a strongly  $M$ - $\alpha$ -reflexive is also strongly  $M$ - $\alpha$ -reflexive. (iii) For any  $i \in I$ , let  $R_i$  be a strongly  $M$ - $\alpha_i$ -reflexive ring with  $\alpha_i$  an endomorphism of  $R_i$ . Then  $S = \prod_{i \in I} R_i$  is strongly  $M$ - $\alpha$ -reflexive, where the endomorphism  $\alpha$  of  $S$  is defined by:

$$\alpha(a_i)_{i \in I} = (\alpha_i(a_i))_{i \in I}.$$

(iv) If  $M = \{e\}$ , then every  $\alpha$ -reflexive ring is strongly  $M$ - $\alpha$ -reflexive, and let  $M = (\mathbb{N} \cup \{\infty\}, +)$ . Thus, a ring  $R$  is strongly  $M$ - $\alpha$ -reflexive if and only if  $R$  is  $\alpha$ -reflexive.

The next example shows that  $\alpha$ -reflexive rings need not be strongly  $M$ - $\alpha$ -reflexive.

**Example 2.1** Let  $R$  be a  $\alpha$ -reflexive ring with an endomorphism  $\alpha$ . Suppose  $S = M_2(R)$ . Then  $S$  is an  $\bar{\alpha}$ -reflexive ring by [5, Theorem 3.3]. However, let

$$\varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} g, \phi = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} g, \psi = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} g$$

be elements in  $S[M]$ . Defined  $\bar{\alpha}$  by  $\bar{\alpha} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ . Then  $\varphi\phi\psi = 0$ , but

$$\psi\phi\bar{\alpha}(\varphi) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} g^2 + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} g^3 \neq 0.$$

Thus  $S$  is not a strongly  $M$ - $\bar{\alpha}$ -reflexive.

A monoid  $M$  is called a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets  $A, B \in M$ , there exists an element  $g \in M$  uniquely in the form of  $ab$  with  $a \in A$  and  $b \in B$ . Every u.p.-monoid  $M$  has no nonunity element of finite order. Let  $(M, \preceq)$  be an ordered monoid. If for any  $g, g', h \in M$ ,  $g \prec g'$  implies that  $gh \prec g'h$  and  $hg \prec hg'$ , then  $(M, \preceq)$  is called a strictly ordered monoid.

**Lemma 2.3**<sup>[9, Lemma 1]</sup> Let  $M$  be a u.p.-monoid and  $R$  a reduced ring. Then  $R[M]$  is reduced.

**Lemma 2.4**<sup>[10, Corollary 2.3]</sup> Let  $M$  be a strictly totally ordered monoid and  $R$  a reduced ring. Then  $R[M]$  is reduced.

**Theorem 2.5** Let  $M$  be a u.p.-monoid and  $R$  a reduced ring. If  $R$  is  $\alpha$ -reflexive, then  $R$  is strongly  $M$ - $\alpha$ -reflexive.

**Proof** Let  $\varphi = a_1 f_1 + \cdots + a_n f_n$  and  $\psi = b_1 h_1 + \cdots + b_m h_m \in R[M]$ , be such that  $\varphi \phi \psi = 0$  for all  $\phi = c_1 g_1 + \cdots + c_k g_k$ . It is obvious that  $R$  is a 3- $M$ -Armendariz ring, since  $R$  is reduced by [13, Proposition 1] and [8, Theorem 2.6]. Thus  $a_i c_k b_j = 0$ , for all  $i, j, k$ . With these facts and the fact that  $R$  is  $\alpha$ -reflexive, we have  $b_j c_k \alpha(a_i) = 0$ . Then  $\psi \phi \alpha(\varphi) = 0$ . This means that  $R$  is strongly  $M$ - $\alpha$ -reflexive.

**Corollary 2.6** Let  $R$  be a 3- $M$ -Armendariz ring. Then  $R$  is  $\alpha$ -reflexive if and only if  $R$  is strongly  $M$ - $\alpha$ -reflexive.

**Corollary 2.7** Let  $M$  be a strictly totally ordered monoid and  $R$  a reduced ring. If  $R$  is  $\alpha$ -reflexive, then  $R$  is strongly  $M$ - $\alpha$ -reflexive.

**Lemma 2.8** Let  $M$  be a u.p.-monoid and  $R$  be a  $\alpha$ -rigid ring. Then  $R[M]$  is  $\alpha$ -rigid.

**Proof** Since  $R$  is  $\alpha$ -rigid if and only if  $R$  is  $\alpha$ -compatible and reduced, thus  $R[M]$  is reduced by Lemma 2.3. We will show that  $R[M]$  is also  $\alpha$ -compatible. Let  $\varphi = \sum_{i=1}^m a_i g_i$ ,  $\psi = \sum_{j=1}^n b_j h_j$  be in  $R[M]$  with  $\varphi \psi = 0$ . Since  $R$  reduced rings are  $M$ -Armendariz rings by [7, Proposition 1.1], we obtain that  $a_i b_j = 0$ . With these facts and the fact that  $R$  is  $\alpha$ -compatible, we have  $a_i \alpha(b_j) = 0$ , thus  $\varphi \alpha(\psi) = 0$ . This means that  $R[M]$  is  $\alpha$ -compatible.

An element  $u$  of a ring  $R$  is called right (resp., left) regular if  $ur = 0$  (resp.,  $ru = 0$ ) implies  $r = 0$  for  $r \in R$ . An element is regular if it is both left and right regular. Recall that a ring  $R$  is called right (resp., left) Ore if given  $a, b \in R$  with  $b$  regular, there exist  $a_1, b_1 \in R$  with  $b_1$  regular such that  $ab_1 = ba_1$  (resp.,  $b_1 a = a_1 b$ ). We suppose that the classical right quotient ring  $Q$  of  $R$  exists. Then for an automorphism  $\alpha$  of  $R$  and any  $ab^{-1} \in Q$  where  $a, b \in R$  with  $b$  regular, the induced map  $\bar{\alpha} : Q \rightarrow Q$  defined by  $\bar{\alpha}(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$  is also an endomorphism. Accordingly, we have the following result.

**Theorem 2.9** Let  $M$  be a u.p.-monoid and  $R$  a right Ore ring with classical right quotient ring  $Q$ . If  $\alpha$  is an epimorphism of  $R$  such that  $R$  is an  $\alpha$ -rigid ring, then  $R$  is strongly  $M$ - $\alpha$ -reflexive if and only if  $Q$  is strongly  $M$ - $\bar{\alpha}$ -reflexive.

**Proof** Assume that  $R$  is strongly  $M$ - $\alpha$ -reflexive. Let  $\varphi = \sum_{i=1}^m u_i f_i$ ,  $\psi = \sum_{j=1}^n v_j g_j$  be in  $Q[M]$  with  $\varphi \phi \psi = 0$  for all  $\phi = \sum_{k=1}^r w_k h_k \in Q[M]$ . Then we have

$$\varphi \phi \psi = \sum_{i=1}^m \sum_{k=1}^r \sum_{j=1}^n (u_i w_k v_j) (f_i h_k g_j) = 0. \quad (2.1)$$

Then by [14, Proposition 2.1.16], we assume that  $u_i = a_i \delta^{-1}$ ,  $v_j = b_j \eta^{-1}$ ,  $w_k = c_k \theta^{-1}$  with  $a_i, b_j, c_k \in R$  for all  $i, j, k$  and some regular elements  $\delta, \eta, \theta \in R$ . Moreover, for each  $k, j$ , there exist  $d_k, e_j \in R$  and regular elements  $\mu, \xi \in R$  such that  $\delta^{-1} c_k = d_k \mu^{-1}$ ,  $(\theta \mu)^{-1} b_j = e_j \xi^{-1}$  also

by [14, Proposition 2.1.16]. Suppose that  $\varphi_1 = \sum_{i=1}^m a_i f_i$ ,  $\psi_1 = \sum_{j=1}^n b_j g_j$ ,  $\phi_1 = \sum_{k=1}^r c_k h_k$ ,  $\psi_2 = \sum_{j=1}^n e_j g_j$  and  $\phi_2 = \sum_{k=1}^r d_k h_k$ . From Eq. (2.1), we have the following equation:

$$\begin{aligned}\varphi\phi\psi &= \sum_{i=1}^m \sum_{k=1}^r \sum_{j=1}^n (a_i \delta^{-1})(c_k \theta^{-1})(b_j \eta^{-1})(f_i h_k g_j) \\ &= \sum_{i=1}^m \sum_{k=1}^r \sum_{j=1}^n a_i d_k e_j (\eta \xi)^{-1} (f_i h_k g_j) \\ &= \varphi_1 \phi_2 \psi_2 (\eta \xi)^{-1} \\ &= 0.\end{aligned}\tag{2.2}$$

Eq. (2.2) implies that  $\varphi_1 \phi_2 \psi_2 = 0$ .

Since  $\varphi_1 \phi_2 \psi_2 = 0$ , we get  $\alpha(\varphi_1 \phi_2 \psi_2) = 0$ , i.e.,  $\alpha(\varphi_1) \alpha(\phi_2 \psi_2) = 0$ . With the fact that  $R$  is  $\alpha$ -rigid if and only if  $R$  is  $\alpha$ -compatible and reduced, we have that  $R[M]$  is  $\alpha$ -compatible and reduced by Lemma 2.8. It follows that  $\alpha(\varphi_1)(\phi_2 \psi_2) = 0$ . Use the condition that  $R[M]$  is reduced and the fact that reduced rings are semicommutative. Therefore,  $\alpha(\varphi_1) \delta \phi_2 \theta \psi_2 = \alpha(\varphi_1) \phi_1 \psi_1 \xi = 0$  and hence we have  $\alpha(\varphi_1) \phi_1 \psi_1 = 0$ . Using [14, Proposition 2.1.16] again, for each  $i, k$ , there exist  $o_i, p_k \in R$  and regular elements  $\sigma, \tau, \varsigma \in R$  such that  $\eta^{-1} c_k = p_k \varsigma^{-1}$  and  $(\theta \varsigma)^{-1} \alpha(a_i) = o_i \tau^{-1}$ . Since  $\alpha$  is an epimorphism of  $R$ , there exists  $q_i \in R$  such that  $o_i = \alpha(q_i)$  for each  $i$ . Suppose that  $\varphi_2 = \sum_{i=1}^m q_i f_i$ ,  $\phi_3 = \sum_{k=1}^r p_k h_k$ , then

$$\begin{aligned}\alpha(\varphi_1) \tau \phi_1 \psi_1 &= \sum_{i=1}^m \sum_{k=1}^r \sum_{j=1}^n \alpha(a_i) \tau c_k b_j (f_i h_k g_j) \\ &= \sum_{i=1}^m \sum_{k=1}^r \sum_{j=1}^n \theta \varsigma \alpha(q_i) c_k b_j (f_i h_k g_j) \\ &= (\theta \varsigma) \alpha(\varphi_2) \phi_1 \psi_1 \\ &= 0.\end{aligned}\tag{2.3}$$

This shows that  $\alpha(\varphi_2) \phi_1 \psi_1 = 0$ . Thus  $\psi_1 \phi_1 \alpha(\alpha(\varphi_2)) = 0$  since  $R$  is strongly  $M$ - $\alpha$ -reflexive. Therefore,  $\psi_1 \phi_1 \alpha(\varphi_2) = 0$ , since  $R[M]$  is an  $\alpha$ -compatible ring. Then

$$\begin{aligned}\psi_1 \eta^{-1} \phi_1 \varsigma \alpha(\varphi_2) &= \sum_{j=1}^n \sum_{k=1}^r \sum_{i=1}^m b_j \eta^{-1} c_k \varsigma \alpha(q_i) (g_j h_k f_i) \\ &= \sum_{j=1}^n \sum_{k=1}^r \sum_{i=1}^m b_j p_k \alpha(q_i) (g_j h_k f_i) \\ &= \psi_1 \phi_3 \alpha(\varphi_2) \\ &= 0\end{aligned}\tag{2.4}$$

Thus, we have

$$\begin{aligned}\psi \phi \bar{\alpha}(\varphi) &= \sum_{j=1}^n \sum_{k=1}^r \sum_{i=1}^m \beta_j \gamma_k \bar{\alpha}(\alpha_i) (g_j h_k f_i) \\ &= \sum_{j=1}^n \sum_{k=1}^r \sum_{i=1}^m b_j p_k \alpha(q_i) \tau^{-1} \alpha(\delta^{-1}) (g_j h_k f_i) \\ &= \psi_1 \phi_3 \alpha(\varphi_2) \tau^{-1} \alpha(\delta^{-1}) \\ &= 0,\end{aligned}\tag{2.5}$$

so that  $Q$  is strongly  $M$ - $\bar{\alpha}$ -reflexive.

**Corollary 2.10** Let  $R$  be a ring and  $\Delta$  a multiplicative monoid in  $R$  consisting of central regular elements. If  $\alpha$  is an automorphism of  $R$ , then  $R$  is strongly  $M$ - $\alpha$ -reflexive if and only if so is  $\Delta^{-1}R$ .

### §3. Extensions of Strongly $M$ - $\alpha$ -reflexive Rings

In this section we examine several kinds of ring extensions which play a very important role in ring theory, being concerned with strongly  $M$ - $\alpha$ -Reflexive rings.

Given a ring  $R$  and a  $(R, R)$ -bimodule  $M$ , the trivial extension of  $R$  by  $M$  is the ring  $T(R, M) = R \oplus M$  with the usual addition and the following multiplication:  $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$ . This is isomorphic to the ring of all matrices  $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$  where  $r \in R$ ,  $m \in M$  and the usual matrix operations are used. For an endomorphism  $\alpha$  of a ring  $R$  and the trivial extension  $T(R, R)$  of  $R$ ,  $\bar{\alpha} : T(R, R) \rightarrow T(R, R)$  defined by  $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$  is an endomorphism of  $T(R, R)$ .

It is easy to see that for an  $\alpha$ -reflexive ring  $R$ ,  $T(R, R)$  need not to be an  $\bar{\alpha}$ -reflexive ring (see [5, Example 2.7(3), Example 3.6]). However, we have the following theorem.

**Theorem 3.1** Let  $M$  be a u.p.-monoid and  $R$  a reduced ring. If  $R$  is an  $\alpha$ -reflexive ring, then  $T(R, R)$  is a strongly  $M$ - $\bar{\alpha}$ -reflexive ring.

**Proof** It is obvious that  $T(R, R)$  is an  $\bar{\alpha}$ -reflexive ring by [4, Proposition 2.2] and a 3- $M$ -Armendariz ring by [8, Corollary 2.15]. It follows that  $T(R, R)$  is a strongly  $M$ - $\bar{\alpha}$ -reflexive ring by Corollary 2.6.

Let  $R$  be a ring. Define a ring  $S_3(R)$  as follows:

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}.$$

**Lemma 3.2** Let  $M$  be a strictly totally ordered monoid and  $R$  a reduced ring. Then  $R$  is a 3- $M$ -Armendariz ring.

**Lemma 3.3** Let  $R$  be a reduced ring. Then  $aRbRb = 0$  if and only if  $aRb = 0$  for  $a, b \in R$ .

**Proof** For  $a, b \in R$ ,  $aRbRb = 0$  implies that  $aRbRbR = 0$  and  $(aRbR)^2 = aRbRaRbR \subseteq aRbRbR = 0$ . Since  $R$  is reduced, then  $aRbR = 0$  and so  $aRb = 0$ . The converse is obvious.

**Theorem 3.4** Let  $M$  be a strictly totally ordered monoid with  $|M| \geq 2$ , and  $R$  a reduced ring. If  $R$  is  $\alpha$ -reflexive ring, then  $S_3(R)$  is strongly  $M$ - $\bar{\alpha}$ -reflexive.

**Proof** It is obvious that  $S_3(R)$  is  $\bar{\alpha}$ -reflexive ring by [5, Theorem 3.3]. Moreover,  $R$  is 3- $M$ -Armendariz by Lemma 3.2, thus  $S_3(R)$  is 3- $M$ -Armendariz by [8, Theorem 2.12]. Then,  $S_3(R)$  is strongly  $M$ - $\bar{\alpha}$ -reflexive by Corollary 2.6.

We denote the  $n \times n$  upper triangular matrix ring over  $R$  by  $T_n(R)$ , respectively. Now we study two special kinds subrings of  $T_n(R)$ . Firstly, define the first special kind subring  $V_n(R)$  of  $T_n(R)$  as follows:

$$V_n(R) = \left\{ \begin{pmatrix} a & 0 & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$

In view of Theorem 3.4, one may suspect that  $V_n(R)$  is strongly  $M$ - $\bar{\alpha}$ -reflexive if  $M$  is a strictly totally ordered monoid and  $R$  is a reduced  $\alpha$ -reflexive ring. But the following example eliminates the possibility.

**Example 3.5** Let  $M$  be a strictly totally ordered monoid and  $R$  a reduced ring. If  $R$  is  $\alpha$ -reflexive, then

$$V_4(R) = \left\{ \begin{pmatrix} a & 0 & b & c \\ 0 & a & d & e \\ 0 & 0 & a & f \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d, e, f \in R \right\}$$

is not strongly  $M$ - $\bar{\alpha}$ -reflexive. In particular, let  $\alpha$  be an endomorphism of  $V_4(R)$  defined by  $\bar{\alpha}(1) = 1$ , for

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, f, \phi = \begin{pmatrix} a & 0 & b & c \\ 0 & a & d & e \\ 0 & 0 & a & f \\ 0 & 0 & 0 & a \end{pmatrix}, g, \psi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad h \in V_4(R)[M],$$

we have  $\varphi\phi\psi = 0$ , but  $\psi\phi\bar{\alpha}(\varphi) \neq 0 \in V_4(R)[M]$ .

Secondly, define  $W_n(R)$  as follows:

$$W_n(R) = \left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 & a_2 \\ 0 & a_1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & a_n \\ 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} \mid a_i \in R, i = 1, \dots, n \right\}.$$

Obviously,  $W_n(R)$  is a subring of  $T_n(R)$ . Then, we have the following result.

**Theorem 3.6** Let  $M$  be a strictly totally ordered monoid and  $R$  a reduced ring. If  $R$  is  $\alpha$ -reflexive, then  $W_n(R)$  is strongly  $M$ - $\bar{\alpha}$ -reflexive.

**Proof** It is obvious that  $R$  is strongly  $M$ - $\alpha$ -reflexive by Corollary 2.7. Then, we use  $(a_1, a_2, \dots, a_n) \in W_n(R)$  to denote

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & a_2 \\ 0 & a_1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & a_n \\ 0 & 0 & \cdots & 0 & a_1 \end{pmatrix},$$

and we can denote their addition and multiplication by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n),$$

$$(a_1, \dots, a_n)(b_1, \dots, b_n) = (a_1 b_1, a_1 b_2 + a_2 b_1, \dots, a_1 b_{n-1} + a_{n-1} b_1, a_1 b_n + a_n b_1).$$

So every elements in  $W_n(R[M])$  can be expressed in the form of  $(\varphi_1, \varphi_2, \dots, \varphi_n)$  for some  $\varphi_i$ 's in  $R[M]$ . Let  $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ ,  $\psi = (\psi_1, \psi_2, \dots, \psi_n) \in W_n(R[M])$  with  $\varphi W_n(R[M])\psi = 0$ . For any  $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in R[M]$ ,  $\varphi\phi\psi = 0$ . Thus we have the following equations:

$$\begin{aligned} \varphi_1 \phi \psi_1 &= 0, \\ \varphi_1 \phi \psi_2 + \varphi_1 \phi_2 \psi_1 + \varphi_2 \phi \psi_1 &= 0, \\ \varphi_1 \phi \psi_3 + \varphi_1 \phi_3 \psi_1 + \varphi_3 \phi \psi_1 &= 0, \\ &\vdots \\ \varphi_1 \phi \psi_{n-1} + \varphi_1 \phi_{n-1} \psi_1 + \varphi_{n-1} \phi \psi_1 &= 0, \\ \varphi_1 \phi \psi_n + \varphi_1 \phi_n \psi_1 + \varphi_n \phi \psi_1 &= 0. \end{aligned}$$

From Eq. (3.1), we see

$$\varphi_1 R[M] \psi_1 = 0, \psi_1 R[M] \alpha(\varphi_1) = 0 \quad (3.1)$$

If we multiply Eq. (3.2) on the right-hand side by  $\tau \psi_1$  for any  $\tau \in R[M]$ , yields  $\varphi_1 \phi \psi_2 \tau \psi_1 + \varphi_2 \phi \psi_1 \tau \psi_1 = 0$  and hence  $\varphi_2 \phi \psi_1 \tau \psi_1 = 0$ , thus  $\varphi_2 R[M] \psi_1 = 0$  by Lemma 3.3 and Eq. (3.6), and  $\varphi_1 R[M] \psi_2 = 0$ . Then

$$\psi_1 R[M] \alpha(\varphi_2) = 0, \psi_2 R[M] \alpha(\varphi_1) = 0. \quad (3.2)$$

If we multiply Eq. (3.3) on the right-hand side by  $\tau \psi_1$  for any  $\tau \in R[M]$ , then  $\varphi_3 R[M] \psi_1 = 0$  and  $\varphi_1 R[M] \psi_3 = 0$  by the similar argument as above. Then

$$\psi_1 R[M] \alpha(\varphi_3) = 0, \psi_3 R[M] \alpha(\varphi_1) = 0. \quad (3.3)$$

Continuing this process, we multiply Eq. (3.5) on the right-hand side by  $\tau \psi_1$  for any  $\tau \in R[M]$ , then  $\varphi_n R[M] \psi_1 = 0$  and  $\varphi_1 R[M] \psi_n = 0$  by the similar argument as above. Then

$$\psi_1 R[M] \alpha(\varphi_n) = 0, \psi_n R[M] \alpha(\varphi_1) = 0.$$



Thus,  $\psi W_n(R[M])\bar{\alpha}(\varphi) = (\psi_1\phi_1\alpha(\varphi_1), \psi_1\phi_1\alpha(\varphi_2) + \psi_1\phi_2\alpha(\varphi_1) + \psi_2\phi_1\alpha(\varphi_1), \dots, \psi_1\phi_1\alpha(\varphi_n) + \psi_1\phi_n\alpha(\varphi_1) + \psi_n\phi_1\alpha(\varphi_1)) = 0$ . Therefore,  $W_n(R)$  is strongly  $M$ - $\bar{\alpha}$ -reflexive.

**Corollary 3.7** Let  $M$  be a strictly totally ordered monoid and  $R$  a reduced ring. If  $R$  is  $\alpha$ -reflexive, then

$$W_3(R) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in R \right\}$$

is strongly  $M$ - $\bar{\alpha}$ -reflexive.

Ali, Elshokry and Liu [10, Proposition 2.24] have shown that if  $R$  is a reduced and strongly  $M$ - $\alpha$ -reversible ring, then  $R[M]$  is a strongly  $N$ - $\alpha$ -reversible ring, where  $M$  and  $N$  are u.p.-monoids. For strongly  $M$ - $\alpha$ -reflexive, we have the following results.

**Proposition 3.8** Let  $M$  and  $N$  be u.p.-monoids. If  $R$  is a reduced and strongly  $M$ - $\alpha$ -reflexive ring, then  $R[M]$  is strongly  $N$ - $\alpha$ -reflexive.

**Proof** It is obvious that  $R[M]$  is reduced by Lemma 2.3, and  $R[M]$  is  $\alpha$ -reflexive since  $R$  is a strongly  $M$ - $\alpha$ -reflexive ring. Thus,  $R[M]$  is strongly  $N$ - $\alpha$ -reflexive by Theorem 2.5.

**Proposition 3.9** Let  $M$  and  $N$  be u.p.-monoids. If  $R$  is a reduced and strongly  $M$ - $\alpha$ -reflexive ring, then  $R[N]$  is strongly  $M$ - $\alpha$ -reflexive.

**Proof** We employ the method of [10, Proposition 2.25]. It is easy to see that there exists an isomorphism of rings  $R[N][M] \cong R[M][N]$  defined by

$$\sum_p \left( \sum_i a_{ip} n_i \right) m_p \rightarrow \sum_i \left( \sum_p a_{ip} m_p \right) n_i.$$

Let  $\varphi_i, \psi_j, \phi_k$  be elements of  $R[N]$  with

$$\left( \sum_i \varphi_i f_i \right) \left( \sum_k \phi_k h_k \right) \left( \sum_j \psi_j g_j \right) = 0,$$

where  $f_i, g_j, h_k \in M$ . Then, we will show that  $\psi_j \phi_k \alpha(\varphi_i) = 0$ . Suppose  $\varphi_i = \sum_p a_{ip} m_p$ ,  $\psi_j = \sum_q b_{jq} n_q$ , and  $\phi_k = \sum_r c_{kr} t_r$ , where  $m_p, n_q, t_r \in N$  for all  $p, q$  and  $r$ . Then we have

$$\left( \sum_i \left( \sum_p a_{ip} m_p \right) f_i \right) \left( \sum_k \left( \sum_r c_{kr} t_r \right) h_k \right) \left( \sum_j \left( \sum_q b_{jq} n_q \right) g_j \right) = 0.$$

Thus,

$$\left( \sum_p \left( \sum_i a_{ip} f_i \right) m_p \right) \left( \sum_r \left( \sum_k c_{kr} h_k \right) t_r \right) \left( \sum_q \left( \sum_j b_{jq} g_j \right) n_q \right) = 0,$$

in  $R[M][N]$ . Consequently, we get

$$\left( \sum_j b_{jq} g_j \right) \left( \sum_k c_{kr} h_k \right) \alpha \left( \left( \sum_i a_{ip} f_i \right) \right) = 0$$

for all  $p, q$  and  $r$ , since  $R[M]$  is strongly  $N$ - $\alpha$ -reflexive by Proposition 3.8. So  $b_{jq}c_{kr}\alpha(a_{ip}) = 0$  for all  $i, j, k, p, q, r$  since  $R$  is strongly  $M$ - $\alpha$ -reflexive. Hence

$$\psi_j \phi_k \alpha(\varphi_i) = \left( \sum_q b_{jq} n_q \right) \left( \sum_r c_{kr} t_r \right) \alpha \left( \left( \sum_p a_{ip} m_p \right) \right) = 0$$

for all  $p, q$  and  $r$ .

Also, in [10], Ali, Elshokry and Liu proved that, for arbitrary monoids  $M$  and  $N$ , if  $R$  is a reduced and strongly  $M$ - $\alpha$ -reversible ring, then  $R$  is strongly  $(M \times N)$ - $\alpha$ -reversible. Similarly, we have the following proposition.

**Proposition 3.10** Let  $M$  and  $N$  be u.p.-monoids. If  $R$  is a reduced and strongly  $M$ - $\alpha$ -reflexive ring, then  $R$  is strongly  $(M \times N)$ - $\alpha$ -reflexive.

**Proof** We use the similar argument to the proof of [10, Proposition 2.26]. Suppose that  $\sum_{i=1}^s a_i(m_i, n_i) \in R[M \times N]$ . Without loss of generality, we assume that  $\{n_1, n_2, \dots, n_s\} = \{n_1, n_2, \dots, n_t\}$  with  $n_i \neq n_j$  when  $1 \leq i \neq j \leq t$ . For any  $1 \leq p \leq t$ , denote  $A_p = \{i | 1 \leq i \leq s, n_i = n_p\}$ . Then  $\sum_{p=1}^t \sum_{i \in A_p} (a_i m_i) n_p \in R[M][N]$ . Note that  $m_i \neq m_{i'}$  for any  $i, i' \in A_p$  with  $i \neq i'$ . Now, it is obvious that there exists an isomorphism of rings  $R[M \times N] \rightarrow R[M][N]$  define by

$$\sum_{i=1}^s a_i(m_i, n_i) \rightarrow \sum_{p=1}^t \sum_{i \in A_p} (a_i m_i) n_p.$$

Let  $\varphi = \sum_{i=1}^s a_i(m_i, n_i)$ ,  $\psi = \sum_{j=1}^{s'} b_j(m'_j, n'_j)$  and  $\phi = \sum_{k=1}^{s''} c_k(m''_k, n''_k)$  be in  $R[M \times N]$  with  $\varphi\psi\phi = 0$ . Thus we have

$$\left( \sum_{i=1}^s a_i(m_i, n_i) \right) \left( \sum_{j=1}^{s'} b_j(m'_j, n'_j) \right) \left( \sum_{k=1}^{s''} c_k(m''_k, n''_k) \right) = 0.$$

Then, we will show that  $\phi\psi\alpha(\varphi) = 0$ . Now, from the above isomorphism it follows that

$$\left( \sum_{p=1}^t \left( \sum_{i \in A_p} a_i m_i \right) n_p \right) \left( \sum_{q=1}^{t'} \left( \sum_{j \in B_q} b_j m'_j \right) n'_q \right) \left( \sum_{r=1}^{t''} \left( \sum_{k \in C_r} c_k m''_k \right) n''_r \right) = 0.$$

Thus

$$\left( \sum_{r=1}^{t''} \left( \sum_{k \in C_r} c_k m''_k \right) n''_r \right) \left( \sum_{q=1}^{t'} \left( \sum_{j \in B_q} b_j m'_j \right) n'_q \right) \alpha \left( \left( \sum_{p=1}^t \left( \sum_{i \in A_p} a_i m_i \right) n_p \right) \right) = 0,$$

since  $R[M]$  is strongly  $M$ - $\alpha$ -reflexive by Proposition 3.8. Note that

$$\begin{aligned} \alpha \left( \sum_{p=1}^t \left( \sum_{i \in A_p} a_i m_i \right) n_p \right) &= \left( \sum_{p=1}^t \alpha \left( \sum_{i \in A_p} a_i m_i \right) n_p \right) \\ &= \left( \sum_{p=1}^t \left( \sum_{i \in A_p} \alpha(a_i) m_i \right) n_p \right). \end{aligned}$$

Then, it is obvious that

$$\left( \sum_{k=1}^{s''} c_k (m''_k, n''_k) \right) \left( \sum_{j=1}^{s'} b_j (m'_j, n'_j) \right) \alpha \left( \left( \sum_{i=1}^s a_i (m_i, n_i) \right) \right) = 0.$$

The ring of Laurent polynomials in  $x$ , with coefficients in a ring  $R$ , consists of all formal sum  $\sum_{i=k}^n m_i x^i$  with obvious addition and multiplication, where  $m_i \in R$  and  $k, n$  are (possibly negative) integers. Denote it by  $R[x; x^{-1}]$ . For an endomorphism  $\alpha$  of  $R$ , we define the map  $\bar{\alpha} : R[x; x^{-1}] \rightarrow R[x; x^{-1}]$  defined by  $\bar{\alpha} \left( \sum_{i=k}^n a_i x^i \right) = \sum_{i=k}^n \alpha(a_i) x^i$  extends  $\alpha$  and also is an endomorphism of  $R[x; x^{-1}]$ . We still denote the map  $\bar{\alpha} : R[x; x^{-1}] \rightarrow R[x; x^{-1}]$  by  $\alpha$ .

The following results extend the class of strongly  $M$ - $\alpha$ -reflexive rings.

**Corollary 3.11** Let  $M$  be a monoid, for a ring  $R$ . If  $\alpha$  is an endomorphism of  $R$ , then  $R[x]$  is strongly  $M$ - $\alpha$ -reflexive if and only if  $R[x; x^{-1}]$  is strongly  $M$ - $\alpha$ -reflexive.

**Proof** It suffices to establish the necessity since  $R[x]$  is a subring of  $R[x; x^{-1}]$ . Suppose that  $R[x]$  is strongly  $M$ - $\alpha$ -reflexive. Let  $\Delta = \{1, x, x^2, \dots\}$ , then clearly  $\Delta$  is a multiplicative closed subset of  $R[x]$ . Since  $R[x; x^{-1}] = \Delta^{-1} R[x]$ , it follows that  $R[x; x^{-1}]$  is strongly  $M$ - $\alpha$ -reflexive by Corollary 2.10.

It was proved in [4, Proposition 3.2] that if  $R$  is an Armendariz ring, then  $R$  is  $\alpha$ -reflexive if and only if  $R[x]$  is  $\alpha$ -reflexive if and only if  $R[x; x^{-1}]$  is  $\alpha$ -reflexive. Also in [10, Proposition 2.17], it was shown that if  $R$  is an  $M$ -Armendariz ring, then  $R$  is strongly right  $M$ - $\alpha$ -reversible if and only if  $R[x; x^{-1}]$  is strongly right  $M$ - $\alpha$ -reversible. Accordingly, we have the equivalence on strongly  $M$ - $\alpha$ -reflexive rings in another situation.

**Proposition 3.12** Let  $M$  be a monoid, and  $R$  a 3- $M$ -Armendariz ring. Then the following are equivalent:

- (1)  $R$  is strongly  $M$ - $\alpha$ -reflexive.
- (2)  $R[x; x^{-1}]$  is strongly  $M$ - $\alpha$ -reflexive.

**Proposition 3.13** Let  $M$  be a monoid, and  $R$  be a ring,  $e'$  a central idempotent of  $R$  with  $\alpha(e') = e'$ . Then the following statements are equivalent:

- (1)  $R$  is strongly  $M$ - $\alpha$ -reflexive.
- (2)  $e'R$  and  $(1 - e')R$  is strongly  $M$ - $\alpha$ -reflexive.

**Proof** (1)  $\Leftrightarrow$  (2) This is straightforward since subrings with  $\alpha(e') = e'$  and finite direct products of strongly  $M$ - $\alpha$ -reflexive rings are strongly  $M$ - $\alpha$ -reflexive.

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