Chin. Quart. J. of Math.
2018, 33 (4): 386—394

On the Restriction Functor of the Relative Stable Category

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Abstract: We prove that, confined that $G > H > P$ and P is a proper p-subgroup of H, if $H \cap {}^gH \leq P$ for any $g \in G - H$, then the operator of the restriction to RH of RG-modules induces a triangulated equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$; if the normal subgroup H controls the fusion of p-subgroups of G, the restriction functor is a faithful triangulated functor; if P is strongly closed in H respect to G, the same functor is also a faithful triangulated functor.

Key words: stable module category; restriction functor; category equivalence
2000 MR Subject Classification: 20C05, 20C20
CLC number: O152.6 **Document code**: A
Article ID: 1002–0462 (2018) 04–0386–09
DOI:10.13371/j.cnki.chin.q.j.m.2018.04.006

§1. Introduction

In representation theory, the stable module category is a category in which projectives are "factored out." The stable module category of the group algebra is a classic example of the triangulated category, it is closed related to the derived category of group algebras and the structure of block algebras. Many authors contribute their ideas in this field([2]∼[3], [6], [8], [12]).

Many ways focal on this category. The stable module category of the group algebra over a field was generalized to the stable module category relative to V-projective modules([5]); over an arbitrary commutative ring however, projective modules are no longer injective, so one needs to adjust this construction, instead, in [1] the authors use the fact that weakly projective modules are weakly injective, where "weakly" means relative to the trivial subgroup, to construct the relative stable category; along this sight, since relative H-projectivity always coincides with

Received date: 2016-11-02
Foundation item: Supported by the National Natural Science Foundation of China(10826057)
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relative H-injectivity for any subgroup H of G, and since the concept of relative projectivity is of fundamental importance in representation theory of finite groups, in section 2, we show the H-relative stable category $\mathrm{Stmod}_H(RG)$, with the method of D. Happel([8]).

Restriction functor is a very natural functor for the module categories of the finite groups, and even for the stable module categories, while for the relative stable module categories, the rules are very different. In section 3, we try to set up this functor, and prove that, confined that $G > H > P$ and P is a proper p-subgroup of H, if $H \cap {}^gH \leq P$ for any $g \in G - H$, then the operator of the restriction to RH of RG-modules induces a triangulated equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$(Theorem 3.1), in addition, a similar result holds for the finite group G with a strongly p-embedded subgroup H(Corollary 3.2), and for the Frobenius group G with the Frobenius complement H(Corollary 3.3); if the normal subgroup H controls the fusion of p-subgroups of G, the restriction functor is a faithful triangulated functor(Theorem 3.6); if P is strongly closed in H respect to G, the same functor is also a faithful triangulated functor(Theorem 3.8).

§2. H-relative Stable Category $\mathrm{Stmod}_H(RG)$

In this paper, we fix a field R of characterize p and a finite group G with $p||G|$; with the H-split short exact sequences of RG-modules, the classical finitely generated RG-module category $\mathrm{mod}(RG)$ is an exact category, and since the relative H-projective RG-module coincides with the relative H-injective RG-module, $\mathrm{mod}(RG)$ is also a Frobenius category([1],[4],[8],[13]). Let $\wp$ be the ideal of the Frobenius category $\mathrm{mod}(RG)$ consisting of all the morphisms factor through the H-projective(H-injective) modules, with $\wp$ we define a quotient category $\mathrm{Stmod}_H(RG)$ as follow

$$\mathrm{Stmod}_H(RG):=\mathrm{mod}(RG)/\wp,$$

where the objects of $\mathrm{Stmod}_H(RG)$ are the same as the objects of $\mathrm{mod}(RG)$, and the morphisms are as follows

$$\mathrm{Hom}_{Stmod_H(RG)}(M,N)=\underline{\mathrm{Hom}}_{RG}(M,N)=\mathrm{Hom}_{RG}(M,N)/\wp(M,N),$$

$M, N \in \mathrm{Stmod}_H(RG)$. $\mathrm{Stmod}_H(RG)$ is called the *H-relative stable category*.

The following result shows that the skeleton of $\mathrm{Stmod}_H(RG)$ is just the full subcategory consisting of the objects which consist no H-projective direct summands.

Lemma 2.1 Let $U, V \in Stmod_H(RG)$, then $U \cong V$ in $\mathrm{Stmod}_H(RG)$ if and only if there exist two $H-$projective RG-modules X and Y such that $U \oplus_R X \cong V \oplus_R Y$ in $\mathrm{mod}(RG)$; in particular, U is $H-$projective in $\mathrm{mod}(RG)$ if and only if U=0 in $\mathrm{Stmod}_H(RG)$.

Proof On the one hand, if $U \cong V$ in $\mathrm{Stmod}_H(RG)$, then there exist $\bar{f}$ and $\bar{g}$ such that $\bar{g}\bar{f} = \bar{1}_U$ and $\bar{f}\bar{g} = \bar{1}_V$, where $\bar{f} \in \underline{\mathrm{Hom}}_{RG}(U,V)$, $\bar{g} \in \underline{\mathrm{Hom}}_{RG}(V,U)$; that is, there exist s, t, h, i such that $1_U - gf = ts$ and $1_V - fg = ih$, where $s \in \mathrm{Hom}_{RG}(U,M)$, $t \in \mathrm{Hom}_{RG}(M,U)$, $h \in$

$\mathrm{Hom}_{RG}(V,N)$, $i\in\mathrm{Hom}_{RG}(N,V)$, M and N are H-projective. We have $1_U=gf+ts$, it means that the following composition of RG-module homomorphisms is 1_U,

$$U\xrightarrow{(f,s)^T}V\oplus_R M\xrightarrow{(g,t)}U,$$

where $(f,s)^T(u)=(f(u),s(u))$, $(g,t)(v,m)=g(v)+t(m)$, $u\in U$, $v\in V$, $m\in M$. Hence, $U|(V\oplus_R M)$, similarly, $V|(U\oplus_R N)$, that is, there exist two H-projective RG-modules X and Y such that $U\oplus_R X\cong V\oplus_R Y$ in $\mathrm{mod}(RG)$.

On the other hand, if there exist two H-projective RG-modules X and Y such that $U\oplus_R X\cong V\oplus_R Y$ in $\mathrm{mod}(RG)$, then there exist $a\in\mathrm{Hom}_{RG}(U\oplus_R X, V\oplus_R Y)$ and $b\in\mathrm{Hom}_{RG}(V\oplus_R Y, U\oplus_R X)$ such that $ba=1_{U\oplus_R X}$ and $ab=1_{V\oplus_R Y}$, where

$$a=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix},\quad b=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix},$$

$a_{11}\in\mathrm{Hom}_{RG}(U,V)$, $a_{12}\in\mathrm{Hom}_{RG}(X,V)$, $a_{21}\in\mathrm{Hom}_{RG}(U,Y)$, $a_{22}\in\mathrm{Hom}_{RG}(X,Y)$, $b_{11}\in\mathrm{Hom}_{RG}(V,U)$, $b_{12}\in\mathrm{Hom}_{RG}(Y,U)$, $b_{21}\in\mathrm{Hom}_{RG}(V,X)$, $b_{22}\in\mathrm{Hom}_{RG}(Y,X)$. So $1_U=b_{11}a_{11}+b_{12}a_{21}$ and $1_V=a_{11}b_{11}+a_{12}b_{21}$, that is, $1_U-b_{11}a_{11}=b_{12}a_{21}$ and $1_V-a_{11}b_{11}=a_{12}b_{21}$, it means that $\bar{1}_U=\bar{b}_{11}\bar{a}_{11}$ and $\bar{1}_V=\bar{a}_{11}\bar{b}_{11}$, and then $U\cong V$ in $\mathrm{Stmod}_H(RG)$.

Remarks 2.2 (1) If H is the trivial subgroup 1 of G, then $\mathrm{Stmod}_H(RG)$ is just the ordinary stable category $\mathrm{Stmod}(RG)$ in the modular representation of finite groups([6],[12]); and it is also a relative stable category modulo weak projective modules ([1]);

(2) If R is the quotient field k of a complete discrete valuation ring, and H is a p-subgroup of G, then $\mathrm{Stmod}_H(RG)$ is the V-relative stable module category $\mathrm{Stmod}_V(kG)$ with $V=Ind_H^G(k)$([5]);

(3) If H contains a Sylow p-subgroup of G, then all the objects of $\mathrm{Stmod}_H(RG)$ are the zero object.

$\mathrm{Stmod}_H(RG)$ is an additive category; moreover, along the method of D. Happel([8]), for the H-split short exact sequences of RG-modules

$$0\to L\to I_k\to T(L,I_k)\to 0,$$

where I_k is H-injective, $k=1,2$. Then, by Schanuel′s Lemma for injective objects in the exact category $\mathrm{mod}(RG)$([9]), $T(L,I_1)\oplus I'\cong T(L,I_2)\oplus I''$ in $\mathrm{mod}(RG)$ with I' and I'' being H-injective(H-projective), and then $T(L,I_1)\cong T(L,I_2)$ in $\mathrm{Stmod}_H(RG)$ by Lemma 2.1. Hence, with $\overline{T}(L):=T(L,I_1)$ we can define an endofunctor $\overline{T}$ of $\mathrm{Stmod}_H(RG)$.

Similarly, with the H-split short exact sequences of RG-module

$$0\to T^{-1}(L,P)\to P\to L\to 0,$$

where P is H-projective, we obtain an endofunctor $\overline{T^{-1}}$ of $\mathrm{Stmod}_H(RG)$ with $\overline{T^{-1}}(L):=T^{-1}(L,P)$.

One can check that $\overline{T}$ and $\overline{T^{-1}}$ are mutually quasi-inverse.

For the H-split short exact sequences of RG-modules

$$0 \to L \xrightarrow{a} M \xrightarrow{b} N \to 0,$$

consider the following commutative diagram in the exact category mod(RG)

$$\begin{array}{ccccccccc} 0 \to & L & \xrightarrow{a} & M & \xrightarrow{b} & N & \to 0 \\ & \| & & \downarrow f & & \downarrow g & \\ 0 \to & L & \to & I & \to & T(L,I) & \to 0, \end{array}$$

where I is H-injective and the rows are H-split short exact sequences. For the above diagram, f exists in mod(RG) since I is H-projective, g is determined uniquely by the chosen f, and then, one can check that $\bar{g}$ is determined uniquely in Stmod$_H(RG)$ independent of f.

Hence, we obtain a standard triangle in Stmod$_H(RG)$ as follow

$$L \xrightarrow{\bar{a}} M \xrightarrow{\bar{b}} N \xrightarrow{\bar{g}} \overline{T}(L). \tag{1}$$

We define the diagram

$$X \to Y \to Z \to \overline{T}(X)$$

in Stmod$_H(RG)$ to be a distinguished triangle if and only if it is isomorphic to a standard triangle, where the suspension functor $\overline{T}$ can be given by taking cosyzygy(with respect to the H-split exact structure in mod(RG)).

With the above structures of triangles, D. Happel′s presentation shows that the quotient category of any Frobenius category is a triangulated category([8, Chapter I.2]). In particular, Stmod$_H(RG)$ is a triangulated category.

§3. The Restriction Functor for Stmod$_H(RG)$

Theorem 3.1 Let $G \geq H > P$ with P being a proper $p-$subgroup of H; if $p \nmid |G : H|$ and $H \cap {}^gH \leq P$ for any $g \in G - H$, then the operator of the restriction to RH of RG-modules induces an equivalence from the triangulated category Stmod$_P(RG)$ to the triangulated category Stmod$_P(RH)$.

Proof The functor is defined as follow

$$Res_H^G(M) := Res_H^G(M), \quad Res_H^G(\bar{f}) := \overline{Res_H^G(f)}$$

for any $M, N \in$ Stmod$_P(RG)$ and $f \in \mathrm{Hom}_{RG}(M, N)$.

(i) If U is a P-projective RG-module, we prove that, $Res_H^G(U)$ here is a P-projective RH-module. It means that the functor Res_H^G is well-defined.

We set $U|Ind_P^G(W)$ for some RP-module W([13, Proposition 11.3.4]), then $Res_H^G(U)|Res_H^G Ind_P^G(W)$, and

$$Res_H^G Ind_P^G(W) = \bigoplus_{g \in [H \backslash G/P]} Ind_{H \cap \, {}^gP}^H({}^gW) \tag{2}$$

by Mackey decomposition formula([13]).

In (2), each $Ind^H_{H\cap\ {}^gP}({}^gW)$ is $(H\cap\ {}^gP)$-projective RH-module, while

$$H\cap\ {}^gP=P,\ g=1;\quad H\cap\ {}^gP\le H\cap\ {}^gH\le P,\ g\neq 1$$

and then each $Ind^H_{H\cap\ {}^gP}({}^gW)$ is a P-projective RH-module, so is $Res^G_H(U)$([13, Proposition 11.3.4]).

It means that, $Res^G_H(U)$ is the zero object of $\mathrm{Stmod}_P(RH)$(Lemma 2.1); and moreover, for any $M,N\in\mathrm{Stmod}_P(RG)$ and $f\in\mathrm{Hom}_{RG}(M,N)$, if $\bar f=0$ in $\underline{\mathrm{Hom}}_{RG}(M,N)$, then $Res^G_H(\bar f)=\overline{Res^G_H(f)}=0$ in $\underline{\mathrm{Hom}}_{RH}(M,N)$. Based on the above, one can check that the functor Res^G_H is well-defined.

(ii) One can check that Res^G_H is an additive functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$, since the functor Res^G_H from $\mathrm{mod}(RG)$ to $\mathrm{mod}(RH)$ is additive.

(iii) For any standard triangle

$$L\xrightarrow{\bar a}M\xrightarrow{\bar b}N\xrightarrow{\bar g}\overline{T}(L)$$

of $\mathrm{Stmod}_P(RG)$ in (1), we see that

$$Res^G_H(L)\xrightarrow{\bar a}Res^G_H(M)\xrightarrow{\bar b}Res^G_H(N)\xrightarrow{\phi\bar g}\overline{T}(Res^G_H(L)),$$

is a distinguished triangle of $\mathrm{Stmod}_P(RH)$ because $\phi: Res^G_H(\overline{T}(L))\to\overline{T}(Res^G_H(L))$ is isomorphic. So the functor Res^G_H is a triangulated (exact) functor.

(iv) Let $M,N\in\mathrm{Stmod}_P(RG)$ and $\bar f\in\underline{\mathrm{Hom}}_{RG}(M,N)$, suppose that $Res^G_H(\bar f)=0$, that is, the restriction of f to H can factor through a P-projective RH-module U, $f=yx$, $x:M\to U$, $y:U\to N$, x and y are RH-module homomorphisms.

We define the map φ and ψ as follows

$$\varphi: M\to Ind^G_H(U),\ m\longmapsto\sum_{g_i\in G/H}g_i\otimes x(g_i^{-1}m),\ m\in M;$$

$$\psi: Ind^G_H(U)\to N,\ \sum_{g_i\in G/H}g_i\otimes u\longmapsto\sum_{g_i\in G/H}g_iy(u),\ u\in U.$$

Then φ and ψ are RG-module homomorphisms, and one can check that $\psi\varphi=|G:H|f$, $f=(\frac{1}{|G:H|}\psi)\varphi$. So f factors through $Ind^G_H(U)$, while $Ind^G_H(U)$ is a P-projective RG-module, $\bar f=0$ in $\mathrm{Stmod}_P(RG)$. The functor Res^G_H is faithful.

(v) Finally, we show that Res^G_H is essentially surjective and full. Let $\omega\in\mathrm{Hom}_{RH}(V,W)$, then

$$\rho: Ind^G_H(V)\to Ind^G_H(W),\ \sum g_i\otimes v_i\longmapsto\sum g_i\otimes\omega(v_i),\ g_i\in[G/H]$$

is an RG-module homomorphism.

$$Res^G_HInd^G_H(V)=\bigoplus_{g\in[H\backslash G/H]}Ind^H_{H\cap\ {}^gH}({}^gV)=V\oplus(\bigoplus_{1\neq g\in[H\backslash G/H]}Ind^H_{H\cap\ {}^gH}({}^gV))\qquad(3)$$

if $g \in G - H$, $H \cap {}^gH \leq P$, each $Ind^H_{H\cap {}^gH}({}^gV)$ in (3) is a P-projective RH-module, and then by Lemma 2.1 $Res^G_H Ind^G_H(V) \cong V$ in $\mathrm{Stmod}_P(RH)$. It means that Res^G_H is essentially surjective.

Similarly, $Res^G_H Ind^G_H(W) \cong W$ in $\mathrm{Stmod}_P(RH)$, and the above two isomorphisms are provided by

$$i : V \to Res^G_H Ind^G_H(V),\ v \longmapsto 1 \otimes v,\ v \in V$$

and

$$p : Res^G_H Ind^G_H(W) \to W,\ \sum g_i \otimes v_i \longmapsto v_1,\ g_1 = 1.$$

$\omega = p \cdot \rho \cdot i$, where i and p are RH-module homomorphisms. So $Res^G_H(\bar{\rho}) \cong \bar{\omega}$, Res^G_H is full.

Summing up the above, Res^G_H is an equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Recall that a proper subgroup H of G is said to be strongly p-embedded in G if $p||H|$ but p does not divide $|H \cap H^g|$ for any $g \in G - H$. In fact, the strongly p-embedded subgroup H of G must contain the normalizer of any non-trivial p-subgroup of G([10]).

Corollary 3.2 Let $G \geq H > P$ with P being a proper p-subgroup of H; if H is a strongly p-embedded in G, then the operator of the restriction to RH of RG-modules induces a triangulated equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof If H is strongly p-embedded in G, then for any $g \in G - H$, $|H \cap {}^gH|$ is a p'-subgroup of G, and $|H \cap {}^gP|$ is also a p'-subgroup of G. Hence, each $Ind^H_{H\cap {}^gP}({}^gV)$ in (2) of the proof of Theorem 3.1 is P-projective, or is 1-projective(weak projective in [1]), that is, is projective relative to the trivial RG-module([13, Proposition 11.3.5]); and each $Ind^H_{H\cap {}^gH}({}^gV)$ in (3) of the proof of Theorem 3.1 is a 1-projective RH-module, and then, each of them is P-projective, too.

It means that, following (i) of the proof of Theorem 3.1, the functor Res^G_H is also well-defined herein, and following (v) of the proof of Theorem 3.1, the functor Res^G_H is essentially surjective and full.

Similarly, following (ii) and (iii) of the proof of Theorem 3.1, the functor Res^G_H is triangulated. Moreover, since H contains the Sylow p-subgroup of G, $p \nmid |G : H|$, hence, following (iv) of the proof of Theorem 3.1, the functor Res^G_H is faithful. So the functor Res^G_H induces a triangulated equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

A finite group G is called a Frobenius group if there exists a non-trivial subgroup H of G such that $H \cap H^g = 1$(the trivial subgroup) for any $g \in G - H$; here H is the so-called Frobenius complement in Frobenius Theorem([15]).

Corollary 3.3 Let G be a Frobenius group with the Frobenius complement H; if P is a proper p-subgroup of H and $p \nmid |G : H|$, then the operator of the restriction to RH of RG-modules induces a triangulated equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof It follows from Theorem 3.1.

Corollary 3.4 Let $G > H > 1$ such that $N_G(K) \leq H$ for any non-trivial subgroup K of

H; if $p \nmid |G:H|$, then for any proper p-subgroup P of H, the operator of the restriction to RH of RG-modules induces a triangulated equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof By [14, Lemma 2.1] we see that G is a Frobenius group with the Frobenius complement H, and H contains a Sylow p-subgroup of G; so the result follows from Corollary 3.3.

Corollary 3.5 Let $G > H > 1$ such that $N_G(K) \leq H$ for any non-trivial subgroup K of H, then for any proper p-subgroup P of H, the operator of the restriction to RH of RG-modules induces a triangulated equivalence from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof Since H contains a Sylow p-subgroup of G, it is true by Corollary 3.4.

We say that a subgroup H of G controls the fusion of p-subgroups of G if H contains a Sylow p-subgroup Q of G, and whenever $P \leq Q$ and $g \in G$ such that ${}^gP \subseteq Q$, there exist $x \in C_G(P)$ and $h \in H$ such that $g = hx$. For example, if H is a subgroup of G with $G = HO_{p'}(G)$, where p is a prime and $O_{p'}(G)$ means the maximal normal p'-subgroup of G, then H controls the fusion of p-subgroups of G([3]).

Theorem 3.6 Let $G \trianglerighteq H$; if $p \nmid |G:H|$ and H controls the fusion of p-subgroups of G, then for any proper p-subgroup P of G, the operator of the restriction to RH of RG-modules induces a faithful triangulated functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof In the case, we note that, G and H have the same Sylow p-subgroups. Moreover, the following Res_H^G is a well-defined functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$:

$$Res_H^G(M) := Res_H^G(M),\ Res_H^G(\bar{f}) = \overline{Res_H^G(f)},$$

where $M, N \in \mathrm{Stmod}_P(RG)$, $f \in \mathrm{Hom}_{RG}(M, N)$.

Indeed, if U is a P-projective RG-module, let $U | Ind_P^G(W)$ for some RP-module W([13, Proposition 11.3.4]), then $Res_H^G(U) | Res_H^G Ind_P^G(W)$, and

$$Res_H^G Ind_P^G(W) = \bigoplus_{g \in [H \backslash G/P]} Ind_{H \cap {}^gP}^H({}^gW) = \bigoplus_{g \in [G/H]} Ind_{H \cap {}^gP}^H({}^gW) \tag{4}$$

by Mackey decomposition formula([13]).

Since H is normal in G and controls the fusion of p-subgroups of G, we see that, H contains a Sylow p-subgroup Q, $H \cap {}^gP = {}^gP \leq {}^gQ \leq H$, and ${}^gQ = {}^hQ$ for some $h \in H$ since ${}^gQ$ is also a Sylow p-subgroup of H. Hence, ${}^{h^{-1}g}Q = Q$, ${}^{h^{-1}g}P \leq Q$, and then $h^{-1}g \in HC_G(P)$, that is, $g \in HC_G(P)$.

From the above, $H \cap {}^gP = {}^gP = {}^hP$ for some $h \in H$, and $Ind_{H \cap {}^gP}^H({}^gW) = Ind_{{}^hP}^H({}^gW)$, it means that each $Ind_{H \cap {}^gP}^H({}^gW)$ in (4) is a ${}^hP$-projective RH-module for each $g \in [G/H]$, and then, it is a P-projective RH-module([7, Theorem 2.7]). Hence,

$$\bigoplus_{g \in [G/H]} Ind_{H \cap {}^gP}^H({}^gW)$$

is a P-projective RH-module, so is the direct summand $Res_H^G(U)$.

It means that, similar as (i) in the proof of Theorem 3.1, Res_H^G is well-defined. And the same as (ii) and (iii) of the proof of Theorem 3.1, it is a triangulated functor; moreover, since $p \nmid |G : H|$, following (iv) in the proof of Theorem 3.1, the functor Res_H^G is also faithful. So Res_H^G is a faithful triangulated functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Corollary 3.7 Let $G \trianglerighteq H$; if H controls the fusion of p-subgroups of G, then for any proper p-subgroup P of G, the operator of the restriction to RH of RG-modules induces a faithful triangulated functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof It follows from Theorem 3.6.

Let $G \geq H \geq K$, we say K is strongly closed in H respect to G if $g^k \in H$ implies that $g^k \in K$, for any $k \in K$ and $g \in G$; in the case, when H is a Sylow p-subgroup of G, K is abbreviated as a strongly closed p-subgroup of G([16]).

Theorem 3.8 Let $G \geq H \geq P$ with P being a proper p-subgroup of H; if $p \nmid |G : H|$ and P is strongly closed in H respect to G, then the operator of the restriction to RH of RG-modules induces a faithful triangulated functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof Firstly, for the functor Res_H^G defined in the proof of Theorem 3.1, now we prove it can be well-defined herein, too.

Indeed, since P is strongly closed in H respect to G, $H \cap {}^gP \leq P$ for any $g \in G$, it means that, each $Ind_{H\cap {}^gP}^H({}^gV)$ in (2) of the proof of Theorem 3.1 is P-projective, hence, following (i) of the proof of Theorem 3.1, the functor Res_H^G is also well-defined herein.

Secondly, similar as (ii),(iii) and (iv) of the proof of Theorem 3.1, the functor Res_H^G is faithful and triangulated. So the functor Res_H^G induces a faithful triangulated functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Corollary 3.9 Let the proper p-subgroup P of G be a strongly closed p-subgroup (in the Sylow p-subgroup H), then the operator of the restriction to RH of RG-modules induces a faithful triangulated functor from $\mathrm{Stmod}_P(RG)$ to $\mathrm{Stmod}_P(RH)$.

Proof It follows from Theorem 3.8.

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