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Construction of Self-dual Codes over $F_p + vF_p$

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Abstract: In this paper, we give an explicit construction for self-dual codes over $F_p + vF_p(v^2 = v)$ and determine all the self-dual codes over $F_p + vF_p$ by using self-dual codes over finite field F_p, where p is a prime.

Key words: Linear code; self-dual code; codes over $F_p + vF_p$; non-chain ring

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§1. Introduction

Codes over finite rings were initiated in the early 1970s [2-3]. They have received much attention after the significant discovery made in the landmark paper [15], which showed that certain good nonlinear binary codes could be found as images of linear codes over $\mathbb{Z}_4$ under the Gray map. Most of the studies are concentrated on the situation in which the ground rings associated with codes are finite chain rings(see, for example, [4, 7, 22, 24-25, 30, 32, 35]). However, Wood proved that finite Frobenius rings are suitable for coding alphabets [31], which leads to many works on non-chain rings. In recent years, linear codes over the ring F_p+vF_p with $v^2 = v$ and p being a prime, which is not a chain ring but a Frobenius ring, have been considered. In [41] Zhu et al. gave some results about cyclic codes over $F_2 + vF_2$, who showed that cyclic codes over the ring are principally generated. In [38], G. Zhang et al. studied cyclic codes with complementary duals over $F_p + vF_p$. In [40] Zhu et al. studied $(1 - 2v)$-constacyclic codes over $F_p + vF_p$, where p is an odd prime. They determined the image of a $(1 - 2v)$-constacyclic code over $F_p + vF_p$ under the Gray map and the structures of such constacyclic codes over $F_p + vF_p$. In [34], G. Zhang studied constacyclic codes over $F_p + vF_p$, and characterized the generator

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polynomials of all constacyclic codes over $F_p + vF_p$ and their dual codes, which generalized the results in [40]. On the other hand, self-dual codes play a very significant role in coding theory both from practical and from theoretical point of view. A vast number of papers have been devoted to the study of self-dual codes, e.g. see ([1], [5]-[6], [8]-[14], [16]-[21], [23], [26]-[27], [29], [33], [37], [39]).

In this paper, we explore the explicit construction of self-dual codes over $F_p + vF_p$ and determine all the self-dual codes over $F_p + vF_p$ in terms of self-dual codes over F_p. Unlike the technique used in the mentioned papers, we give the characterization of the torsion codes associated with the linear codes and their duals over $F_p + vF_p$. They are used as a tool to study self-dual codes over $F_p + vF_p$ and their explicit construction.

§2. Preliminaries

Let F_p be a finite field with p elements, where p is a prime. Throughout this paper, let R be the commutative ring $F_p + vF_p = \{a + vb | a, b \in F_p\}$, where $v^2 = v$. The ring R is a semi-local ring with two maximal ideals given by $\langle v\rangle = \{av | a \in F_p\}$ and $\langle 1 - v\rangle = \{a(1 - v) | a \in F_p\}$. It is easy to verify that both $R/\langle v\rangle$ and $R/\langle 1 - v\rangle$ are isomorphic to F_p. Any element of R can be expressed as $c = a + vb$, where $a, b \in F_p$. The Gray map Φ from R to $F_p \oplus F_p$ is given by $\Phi(c) = (a, a + b)$, thus Φ is a ring isomorphism, which means that R is isomorphic to the ring $F_p \oplus F_p$. Therefore R is a finite Frobenius ring.

A linear code of length n over R is an R-submodule of R^n, where

$$R^n = \{(r_1, r_2, \cdots, r_n) \mid r_i \in R, \forall 1 \le i \le n\}.$$

For $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in R^n$, the inner product $x \cdot y$ of x, y is defined as $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Let C be a linear code of length n over R. We define $C^{\perp} = \{x \in R^n \mid x \cdot c = 0, \forall c \in C\}$ to be the orthogonal code of C, which is called the *dual code* of C. Note that $C^{\perp}$ is also a linear code. If $C = C^{\perp}$, then C is called *self-dual*.

Note that any element c of R^n can be expressed as $c = a + vb$, where $a, b \in F_p^n$. Define C_1 and C_2 as follows:

$$C_1 = \{a \in F_p^n | a + vb \in C, \text{ for some } b \in F_p^n\};\ \ C_2 = \{a + b \in F_p^n | a + vb \in C\}.$$

Obviously, C_1 and C_2 are linear codes over F_p.

We know that the ring R has two maximal ideals $\langle v\rangle$ and $\langle 1 - v\rangle$. Their residue fields are both F_p. Thus we have two canonical projections defined as follows:

$$R = F_p + vF_p \longrightarrow R/\langle 1 - v\rangle = F_p : r + vq \longmapsto r + q;$$

and

$$R = F_p + vF_p \longrightarrow R/\langle v\rangle = F_p : r + vq \longmapsto r.$$

We simple denote these two projections by "$\widehat{\ }$" and "$-$", respectively. Denote by $\widehat{r}$ and $\overline{r}$ the images of an element $r \in R$ under these two projections, respectively.

For a code C of length n over R, let $a \in R$. The submodule quotient is a linear code of length n over R, defined as follows:

$$(C : a) = \{x \in R^n | ax \in C\}.$$

The codes $\widehat{(C : v)}$ and $\overline{(C : (1-v))}$ over the field F_p is called the *torsion codes* associated with the code C over the ring R.

§3. Self-dual Codes Over $F_p + vF_p$

In this section we will determine all the self-dual codes over R in terms of those over F_p. With notation as above, the following results are very useful.

lemma 3.1[36] Let C be a linear code of length n over R. Then

(1) $\widehat{(C : v)} = C_2$;

(2) $\overline{(C : (1-v))} = C_1$;

(3) $(\widehat{(C : v)})^{\perp} = \widehat{(C^{\perp} : v)}$; $(\overline{(C : (1-v))})^{\perp} = \overline{(C^{\perp} : (1-v))}$.

Let A, B be the codes over R. We denote that $A \oplus B = \{a + b | a \in A, b \in B\}$.

lemma 3.2[36] Let C be a linear code of length n over R. Then C can be uniquely expressed as $C = vC_2 \oplus (1-v)C_1$. Moreover, we also have $C^{\perp} = vC_2^{\perp} \oplus (1-v)C_1^{\perp}$.

Theorem 3.3 Let C be a linear code of length n over R. Then C is a self-dual code if and only if C_1 and C_2 are both self-dual codes.

Proof ($\Longrightarrow$) Let C be a self-dual code. Then by Lemma 3.1 we have that

$$C_1^{\perp} = (\overline{(C : (1-v))})^{\perp} = \overline{(C^{\perp} : (1-v))} = \overline{(C : (1-v))} = C_1$$

and

$$C_2^{\perp} = (\widehat{(C : v)})^{\perp} = \widehat{(C^{\perp} : v)} = \widehat{(C : v)} = C_2,$$

that is, C_1 and C_2 are both self-dual codes.

($\Longleftarrow$) Let C_1 and C_2 are both self-dual codes. Then by Lemma 3.2

$$C^{\perp} = vC_2^{\perp} \oplus (1-v)C_1^{\perp} = vC_2 \oplus (1-v)C_1 = C.$$

So C is self-dual.

Remark 3.4 According to Lemma 3.2 and Theorem 3.3, it is clear that a self-dual code over R can be explicitly expressed by some two self-dual codes over F_p. We need to study the converse part, which is a crucial and interesting step.

§4. Construction for Self-dual Codes Over $F_p + vF_p$

The construction of self-dual codes over R depends on the following theorem.

Theorem 4.1 Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear codes of length n over F_p with generator matrices G_1 and G_2, respectively. Let l_1 and l_2 be the dimensions of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then the linear code C over R generated by the matrix G satisfies

$$\widehat{(C:v)} = \mathcal{C}_2;\ \ \overline{(C:(1-v))} = \mathcal{C}_1$$

and $C = v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1$, where

$$G = \begin{cases} \begin{pmatrix} vG_2 \\ 0 \end{pmatrix} + (1-v)G_1, & \text{if } l_1 > l_2; \\ vG_2 + \begin{pmatrix} (1-v)G_1 \\ 0 \end{pmatrix}, & \text{if } l_1 < l_2; \\ vG_2 + (1-v)G_1, & \text{if } l_1 = l_2. \end{cases}$$

Proof We only prove the case $l_1 > l_2$, as the proof of the other cases is similar to this one. Assume that

$$G_1 = \begin{pmatrix} g_{11} \\ g_{12} \\ \vdots \\ g_{1l_1} \end{pmatrix}; G_2 = \begin{pmatrix} g_{21} \\ g_{22} \\ \vdots \\ g_{2l_2} \end{pmatrix},$$

where $g_{1i}(\forall 1 \le i \le l_1)$ and $g_{2j}(\forall 1 \le j \le l_2)$ are row vectors of G_1 and G_2, respectively. Then

$$G = \begin{pmatrix} vg_{21} + (1-v)g_{11} \\ vg_{22} + (1-v)g_{12} \\ \vdots \\ vg_{2l_2} + (1-v)g_{1l_2} \\ (1-v)g_{1,l_2+1} \\ \vdots \\ (1-v)g_{1l_1} \end{pmatrix}.$$

Since $vg_{2i} + (1-v)g_{1i} \in C$, i.e., $g_{1i} + v(g_{2i} - g_{1i}) \in C$, for $1 \le i \le l_2$, by Lemma 3.1 we have

$$g_{2i} = g_{1i} + (g_{2i} - g_{1i}) \in \widehat{(C:v)},$$

for $1 \le i \le l_2$. Therefore $\mathcal{C}_2 \subseteq \widehat{(C:v)}$.

Let $y \in \widehat{(C:v)}$, then there exists $x \in (C:v)$ such that $y = \widehat{x}$. Since $vx \in C$, we may assume that

$$vx = \sum_{i=1}^{l_2}(a_i + vs_i)[vg_{2i} + (1-v)g_{1i}] + \sum_{l_2+1}^{l_1}(a_i + vs_i)[(1-v)g_{1i}],$$

where $a_i + vs_i \in F_p + vF_p$, for $1 \le i \le l_1$. So

$$vx = v^2x = v \cdot vx = v\sum_{i=1}^{l_2}(a_i + s_i)g_{2i}.$$

Let $x = x_1 + vx_2, x_1, x_2 \in F_p^n$. Then $\widehat{x} = x_1 + x_2$. Thus

$$v(x_1 + x_2) = v(x_1 + vx_2) = vx = v\sum_{i=1}^{l_2}(a_i + s_i)g_{2i}.$$

Hence $x_1 + x_2 = \sum_{i=1}^{l_2}(a_i + s_i)g_{2i}$. Therefore

$$y = \widehat{x} = x_1 + x_2 = \sum_{i=1}^{l_2}(a_i + s_i)g_{2i} \in \mathcal{C}_2,$$

by which we obtain $\widehat{(C : v)} \subseteq \mathcal{C}_2$. From the above results we get that $\widehat{(C : v)} = \mathcal{C}_2$.

On the other hand, note that

$$vg_{2i} + (1 - v)g_{1i} \in C, i.e.\ \ g_{1i} + v(g_{2i} - g_{1i}) \in C,$$

for $1 \le i \le l_1$, where $g_{2i} = 0$, if $i > l_2$. By Lemma 3.1 we have $g_{1i} \in \overline{(C : (1 - v))}$, for $1 \le i \le l_1$. Therefore $\mathcal{C}_1 \subseteq \overline{(C : (1 - v))}$.

Let $z \in \overline{(C : (1 - v))}$, then there exists $s \in (C : (1 - v))$ such that $z = \overline{s}$. Since $(1 - v)s \in C$, we may assume that

$$(1 - v)s = \sum_{i=1}^{l_2}(b_i + vt_i)[vg_{2i} + (1 - v)g_{1i}] + \sum_{l_2+1}^{l_1}(b_i + vt_i)[(1 - v)g_{1i}],$$

where $b_i + vt_i \in F_p + vF_p$, for $1 \le i \le l_1$. So

$$(1 - v)s = (1 - v)^2s = (1 - v) \cdot (1 - v)s = (1 - v)\sum_{i=1}^{l_1} b_ig_{1i}.$$

Let $s = s_1 + vs_2, s_1, s_2 \in F_p^n$. Then $\overline{s} = s_1$. Thus

$$(1 - v)s_1 = (1 - v)(s_1 + vs_2) = (1 - v)s = (1 - v)\sum_{i=1}^{l_1} b_ig_{1i}.$$

Hence $s_1 = \sum_{i=1}^{l_1} b_ig_{1i}$. Therefore we have

$$z = \overline{s} = s_1 = \sum_{i=1}^{l_1} b_ig_{1i} \in \mathcal{C}_1,$$

which implies that $\overline{(C : (1 - v))} \subseteq \mathcal{C}_1$. Thus we get that $\overline{(C : (1 - v))} = \mathcal{C}_1$.

Finally, by Lemma 3.2 and Lemma 3.2,

$$C = v\widehat{(C : v)} \oplus (1 - v)\overline{(C : (1 - v))}$$

$$= v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1,$$

which is our desired result. Thus we complete the proof.

Corollary 4.2 Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two self-dual codes of length n over F_p with generator matrices G_1 and G_2, respectively, then the linear code C over R generated by the matrix G is also self-dual, where

$$G = vG_2 + (1-v)G_1,$$

and $C = v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1$.

Proof Noting that $l_1 = l_2$ in this case, by Lemma 3.1 and Lemma 3.2 and Theorem 4.1 we have that

$$\begin{aligned} C^{\perp} &= v(\widehat{(C:v)})^{\perp} \oplus (1-v)(\overline{(C:(1-v))})^{\perp} \\ &= v\mathcal{C}_2^{\perp} \oplus (1-v)\mathcal{C}_1^{\perp} \\ &= v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1 \\ &= C. \end{aligned}$$

So C is self-dual.

Remark 4.3 We mention that one can check in the same way as in Theorem 4.1 and Corollary 4.2 that all the results also hold for the case when G_1 and G_2 are two generating matrices, i.e., they are only matrices that generate $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We reformulate the result as follows.

Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear codes of length n over F_p with generating matrices G_1 and G_2, respectively, and let r_1 and r_2 be the numbers of the rows of G_1 and G_2, respectively, then the code C over R generated by the matrix G as follows satisfies

$$\widehat{(C:v)} = \mathcal{C}_2; \ \ \overline{(C:(1-v))} = \mathcal{C}_1$$

and $C = v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1$, where

$$G = \begin{cases} \begin{pmatrix} vG_2 \\ 0 \end{pmatrix} + (1-v)G_1, & \text{if } r_1 > r_2; \\ vG_2 + \begin{pmatrix} (1-v)G_1 \\ 0 \end{pmatrix}, & \text{if } r_1 < r_2; \\ vG_2 + (1-v)G_1, & \text{if } r_1 = r_2. \end{cases}$$

On the other hand, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are self-dual, then the code C generated by the above matrix G is also self-dual.

Theorem 4.4 All the self-dual codes over R are given by

$$v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1,$$

where $\mathcal{C}_1, \mathcal{C}_2$ range over all the self-dual codes over F_p, respectively. And this expression is unique, i.e., if

$$v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1 = v\mathcal{C}_2' \oplus (1-v)\mathcal{C}_1',$$

then $\mathcal{C}_2 = \mathcal{C}_2'$ and $\mathcal{C}_1 = \mathcal{C}_1'$, where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_1'$ and $\mathcal{C}_2'$ are all self-dual codes over F_p.

Proof First according to Remark 3.4, every self-dual code over R can be explicitly expressed by two fixed self-dual codes over F_p as in the above form.

Next, let $\mathcal{C}_1, \mathcal{C}_2$ be two self-dual codes over F_p. Assume that G_1 and G_2 are generator matrices for $\mathcal{C}_1, \mathcal{C}_2$, respectively. Then according to Corollary 4.2 we know that the code C generated by the matrix $vG_1 + (1-v)G_2$ is self-dual and satisfies $C = v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1$.

Let $x \in \mathcal{C}_2$. Since $v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1 = v\mathcal{C}_2' \oplus (1-v)\mathcal{C}_1'$, we have that

$$vx \in v\mathcal{C}_2 \subseteq v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1 = v\mathcal{C}_2' \oplus (1-v)\mathcal{C}_1'.$$

Setting $vx = vx' + (1-v)y'$, where $x' \in \mathcal{C}_2', y' \in \mathcal{C}_1'$, we get that $v(x-x') = (1-v)y'$ and $v(x-x') = 0$, so $x = x'$. Therefore $\mathcal{C}_2 \subseteq \mathcal{C}_2'$. Similarly, we have $\mathcal{C}_2' \subseteq \mathcal{C}_2$. Hence $\mathcal{C}_2 = \mathcal{C}_2'$.

Let $z \in \mathcal{C}_1$. Since $v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1 = v\mathcal{C}_2' \oplus (1-v)\mathcal{C}_1'$, we have that

$$(1-v)z \in (1-v)\mathcal{C}_1 \subseteq v\mathcal{C}_2 \oplus (1-v)\mathcal{C}_1 = v\mathcal{C}_2' \oplus (1-v)\mathcal{C}_1'.$$

Setting $(1-v)z = va' + (1-v)z'$, where $a' \in \mathcal{C}_2', z' \in \mathcal{C}_1'$, we get that $(1-v)(z-z') = va'$ and $(1-v)(z-z') = 0$, so $z = z'$. Therefore $\mathcal{C}_1 \subseteq \mathcal{C}_1'$. Similarly, we have $\mathcal{C}_1' \subseteq \mathcal{C}_1$. Hence $\mathcal{C}_1 = \mathcal{C}_1'$. Thus we complete the proof.

Corollary 4.5 Let $N(R)$ be the number of self-dual codes of length n over R and $N(F_p)$ the number of self-dual codes of length n over F_p. Then

$$N(R) = N(F_p)^2.$$

Proof It follows from Theorem 4.4.

The following lemma is well known and can be found in [28].

Lemma 4.6 Let F_q be a finite field with characteristic p. Then

(i) If $p = 2$ or $p \equiv 1 \pmod 4$, then a self-dual code of length n exists over F_q if and only if $n \equiv 0 \pmod 2$.

(ii) If $p \equiv 3 \pmod 4$, then a self-dual code of length n exists over F_q if and only if $n \equiv 0 \pmod 4$.

Now Combining Theorem 4.4 with Lemma 4.6, the following result is easily obtained.

Theorem 4.7 With the above notation. Then the following holds:

(i) If $p = 2$ or $p \equiv 1 \pmod 4$, then a self-dual code of length n over R exists if and only if $n \equiv 0 \pmod 2$.

(ii) If $p \equiv 3 \pmod 4$, then a self-dual code of length n over R exists if and only if $n \equiv 0 \pmod 4$.

§5. Examples

According to Corollary 4.2, the construction of self-dual codes over R hinges on constructing the self-dual codes over F_p. See [19] on the building-up construction for self-dual codes over F_p. The following examples illustrate the result.

Example 5.1 Consider the construction of self-dual code of length 6 over $R = F_2 + vF_2$. Here $l_1 = l_2 = 3$ and

$$G_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}; G_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then the code C of length 6 over $R = F_2 + vF_2$ generated by the following matrix

$$\begin{aligned} G &= vG_2 + (1-v)G_1 = G_1 + v(G_2 - G_1) \\ &= \begin{pmatrix} 1 & 0 & 1+v & 1 & 0 & 1+v \\ 1+v & 1+v & 1 & 0 & 1+v & v \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

is self-dual.

Example 5.2 Consider the construction of self-dual code of length 12 over $R = F_3 + vF_3$. Here $l_1 = l_2 = 6$ and

$$G_1 = (I_6 \mid B),$$

where I_6 denotes the 6×6 identity matrix, and

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 0 \end{pmatrix},$$

i.e., the code with generator matrix G_1 is the ternary Golay code;

$$G_2 = \begin{pmatrix} 0\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,1\,0\,1\,2\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,1\,1\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,1\,0\,2\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,2\,1\,0 \\ 2\,1\,2\,0\,1\,2\,1\,0\,2\,2\,0\,1 \end{pmatrix}.$$

Then the code C of length 12 over $R = F_3 + vF_3$ generated by the following matrix

$$G = vG_2 + (1-v)G_1 = G_1 + v(G_2 - G_1) =$$

$$\begin{pmatrix} 1+2v & v & v & v & 0 & 0 & 0 & 1+2v & 1+2v & 1+2v & 1+2v & 1+2v \\ v & 1+2v & 0 & 0 & v & 0 & 1 & 2v & 1+2v & 2+2v & 2+2v & 1+2v \\ 0 & 0 & 1+2v & 0 & 0 & v & 1 & 1 & 0 & 1+2v & 2+v & 2+v \\ 0 & 0 & 0 & 1+2v & v & 0 & 1+2v & 2+v & 1 & 0 & 1+v & 2+v \\ 0 & 0 & 0 & 0 & 1+2v & 0 & 1+2v & 2+v & 2+2v & 1+v & v & 1+2v \\ 2v & v & 2v & 0 & v & 1+v & 1 & 1+2v & 2 & 2 & 1+2v & v \end{pmatrix}.$$

is self-dual.

§6. Conclusion

In this paper, we completely determine the construction and enumeration of self-dual codes over $F_p + vF_p$, which is a non-chain ring. Now we leave the reader with the possible direction for further work. It is known that any nonzero constacyclic code over $F_p + vF_p$ has a unique generating set in standard form. So it is nature and would be interesting to study the self-dual constacyclic codes over finite non-chain rings in terms of their generating set in standard form.

[References]

[1] ALFARO R, DHUL-QARNAYN K. Constructing self-dual codes over $F_q[u]/(u^t)$[J]. Des Codes Cryptogr, 2015, 74(2): 453-465.

[2] BLAKE I F. Codes over certain rings[J]. Inform Control, 1972, 20: 396-404.

[3] BLAKE I F. Codes over integer residue rings[J]. Inform Control, 1975, 29: 295-300.

[4] CHEN Bo-cong, LING San, ZHANG Guang-hui. Enumeration formulas for self-dualcyclic codes[J]. Finite Fields Appl, 2016, 42: 1-22.

[5] DOUGHERTY S T, FERNÁNDEZ-CÓRDOBA C. Z_2Z_4-additive formally self-dual codes[J]. Des Codes Cryptogr, 2014, 72(2): 435-453.

[6] DINH Hai Q, FAN Yun, LIU Hua-lu, LIU Xiu-sheng, SRIBOONCHITTA Songsak. On self-dual constacyclic codes of length ps over $F_{p^m} + uF_{p^m}$[J]. Discrete Math, 2018, 341(2): 324-335.

[7] DINH H Q, LÓPEZ-PERMOUTH S R. Cyclic and negacyclic codes over finite chain rings[J]. IEEE Trans Inform Theory, 2004, 50(8): 1728-1744.

[8] DOUGHERTY S T, HARADA M, SOLÉ P. Self-dual codes over rings and the Chinese remainder theorem[J]. Hokkaido Math J, 1999, 28: 253-283.

[9] DOUGHERTY Steven T, KIM Jon-Lark, KULOSMAN Hamid, LIU Hong-wei. Self-dualcodes over commutative Frobenius rings[J]. Finite Fields Appl, 2010, 16: 14-26.

[10] DOUGHERTY S T, KIM J -L, LIU H. Constructions of self-dual codes over finite commutative chain rings[J]. Int J Inform Coding Theory, 2010, 1(2): 171-190.

[11] DOUGHERTY Steven T, KAYA Abidin, SALTÜRK Esengl. Constructions of self-dualcodes and formally self-dual codes over rings[J]. Appl Algebra Engrg Comm Comput, 2016, 27(5): 435-449.

[12] DOUGHERTY Steven T, LEROY Andr, Euclidean self-dual codes over non-commutative Frobenius rings[J]. Appl Algebra Engrg Comm Comput, 2016, 27(3): 185-203.

[13] GILDEA Joe, KAYA Abidin, TAYLOR Rhian, YILDIZ Bahattin. Constructions for self-dual codes induced from group rings[J]. Finite Fields Appl, 2018, 51: 71-92.

[14] HAN Sunghyu. A method for constructing self-dual codes over Z_{2^m}[J]. Des Codes Cryptogr, 2015, 75(2): 253-262.

[15] HAMMONS A R, JR., KUMAR P V, CALDERBANK A R, SLOANE N J A, SOLÉ P. The $\mathbb{Z}_4$ linearity of Kerdock, Preparata, Goethals and related codes[J]. IEEE Trans Inform Theory, 1994, 40(2): 301-319.

[16] JITMAN Somphong, LING San, SANGWISUT Ekkasit. On self-dual cyclic codes of length p^a over $GR(p^2, s)$[J]. Adv Math Commun, 2016, 10(2): 255-273.

[17] KARADENIZ Suat, DOUGHERTY Steven T, YILDIZ Bahattin. Constructing formally self-dual codes over R_k[J]. Discrete Appl Math, 2014, 167: 188-196.

[18] KIM Boran, LEE Yoonjin. Lee weights of cyclic self-dual codes over Galois rings of characteristic p^2[J]. Finite Fields Appl, 2017, 45: 107-130.

[19] KIM J -L, LEE Y. Euclidean and Hermitian self-dual MDS codes over large finite fields[J]. J Combin Theory, Ser A, 2004, 105: 79-95.

[20] KIM J -L, LEE Y. Construction of MDS self-dual codes over Galois rings[J]. Des Codes Cryptogr, 2007, 45(2): 247-258.

[21] KAYA Abidin, YILDIZ Bahattin. Various constructions for self-dual codes over rings and new binary self-dual codes[J]. Discrete Math, 2016, 339(2): 460-469.

[22] KAI Xiao-shan, ZHU Shi-xin. Negacyclic self-dual codes over finite chain rings[J]. Des Codes Cryptogr, 2012, 62(2): 161-174.

[23] KAI Xiao-shan, ZHU Shi-xin, TANG Yong-sheng. Some constacyclic self-dual codesover the integers modulo 2^m[J]. Finite Fields Appl, 2012, 18(2): 258-270.

[24] LING San, BLACKFORD J. $\mathbb{Z}_{p^{k+1}}$-linear codes[J]. IEEE Trans Inform Theory, 2002, 48: 2592-2605.

[25] LIU Xiu-sheng, LIU Hua-lu. Cyclic codes over $F_2 + uF_2 + vF_2$[J]. Chin Quart J of Math, 2014, 29(2): 189-194.

[26] LIU Xiu-sheng, LIU Hua-lu. The self-dual codes over formal power series rings[J]. Chin Quart J of Math, 2014, 29(1): 30-38.

[27] NAGATA Kiyoshi, NEMENZO Fidel, WADA Hideo. Mass formula and structure of self-dual codes over Z_{2^s}[J]. Des Codes Cryptogr, 2013, 67(3): 293-316.

[28] RAINS E, SLOANE N J A. Self-dual codes, in the Handbook of Coding Theory, V. S. Pless and W. C[J]. Huffman, eds., Elsevier, Amsterdam, 1998, 177-294.

[29] SHARMA Anuradha, SHARMA Amit K. Construction of self-dual codes over Z_{2^m}[J]. Cryptogr Commun, 2016, 8(1): 83-101.

[30] WOLFMANN J. Binary image of cyclic codes over $\mathbb{Z}_4$[J]. IEEE Trans Inform Theory, 2001, 47(5): 1773-1779.

[31] WOOD J. Duality for modules over finite rings and applications to coding theory[J]. Amer J Math, 1999, 121: 555-575.

[32] WU Yan-sheng, YUE Qin, LI Feng-wei. Primitive idempotents of irreducible cyclic codes and self-dual cyclic codes over Galois rings[J]. Discrete Math, 2018, 341(6): 1755-1767.

[33] YOO Jinjoo, LEE Yoonjin, KIM Boreum. Constructions of formally self-dual codesover Z_4 and their weight enumerators[J]. IEEE Trans Inform Theory, 2017, 63(12): 7667-7675.

[34] ZHANG Guang-hui. Constacyclic codes over $F_p + vF_p$[J]. Ars Comb, 2018, 139: 55-67.

[35] ZHANG Guang-hui. A note on self-dual cyclic codes over finite chain rings[J]. Adv Math (China), 2016,

45(2): 206-210.

[36] ZHANG Guang-hui, LI Liang-chen. Self-dual codes and cyclic codes over $F_p + vF_p$[J]. Ars Comb, 2014, 116: 445-455.

[37] ZHANG Guang-hui, LIU Hong-wei. Self-dual permutation codes over formal power series rings and finite principal ideal rings[J]. Acta Math Sci Ser B Engl Ed, 2013, 33(6): 1695-1710.

[38] ZHANG Guang-hui, LIU Qing-qing. Cyclic codes with complementary duals over $F_p + vF_p$[J]. Chin Quart J of Math, 2016, 31(2): 118-124.

[39] ZHU Shi-xin, QIAN Kai-yan, KAI Xiao-shan. A note on negacyclic self-dual codesover Z_{2^a}[J]. Discrete Math, 2012, 312(22): 3270-3275.

[40] ZHU Shi-xin, WANG Li-qi. A class of constacyclic codes over $F_p + vF_p$ and its Gray image[J]. Discrete Math, 2011, 311: 2677-2682.

[41] ZHU Shi-xin, WANG Yu, SHI Min-jia. Some results on cyclic codes over $F_2 + vF_2$[J]. IEEE Trans Inform Theory, 2010, 56(4): 1680-1684.