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A Characterization of The Twisted Heisenberg-Virasoro Vertex Operator Algebra

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Abstract: The twisted Heisenberg-Virasoro algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one. In this paper, we first study the variety of semi-conformal vectors of the twisted Heisenberg-Virasoro vertex operator algebra, which is a finite set consisting of two nontrivial elements. Based on this property, we also show that the twisted Heisenberg-Virasoro vertex operator algebra is a tensor product of two vertex operator algebras. Moreover, associating to properties of semi-conformal vectors of the twisted Heisenberg-Virasoro vertex operator algebra, we charaterized twisted Heisenberg-Virasoro vertex operator algebras. This will be used to understand the classification problems of vertex operator algebras whose varieties of semi-conformal vectors are finite sets.

Key words: Twisted Heisenberg-Virasoro algebra; Vertex operator algebra; Semi-conformal vector; Semi-conformal subalgebra

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§1. Introduction

The twisted Heisenberg-Virasoro algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one, which has been first studied by Arbarello etal, in Ref. [3]. It contains the classical Heisenberg algebra and the Virasoro algebra as subalgebras. And they also have established a connection between the second cohomology

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of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one.

The representation theory of the twisted Heisenberg-Virasoro algebra is closely related to those of other Lie algebras, such as the Virasoro algebra and toroidal Lie algebras, and has been studied in Refs. [1, 4-6, 14, 19-20, 26-27]. In Ref.[4], the free field realization of the twisted Heisenberg-Virasoro algebra at level zero is given and its applications can be obtained. In Ref.[2], A. Alexandrov constructed new relations connecting Kontsevich-Witten tau-functions, Hodge integrals and Hurwitz numbers and derived linear constraints for all of them. These constraints as operators form a twisted Heisenberg-Virasoro algebra.

It's well known that the vertex operator algebra theory provides a rigorous mathematical foundation for two dimensional conformal field theory and string theory from the Hamiltonian point of view(Refs.[23, 28]). It follows from Proposition 3.1 in Ref.[6] that the twisted Heisenberg-Virasoro vertex operator algebra has a vertex operator algebra structure which is the tensor product of a Virasoro vertex operator algebra and a Heisenberg vertex operator algebra. In Refs.[8-9], we used their semi-conformal vectors to describe Heisenberg vertex operator algebras and affine vertex operator algebras (began from Refs.[21-22]). [8, Theorem 1.1] tells that the set of all semi-conformal vectors of a vertex operator algebra $V=\oplus_{n\in\mathbb{Z}}V_n$ forms a Zarisk closed subset (or, an affine algebraic variety) in the weight-two subspace V_2 . For the twisted Heisenberg-Virasoro vertex operator algebra, we find it has only two nontrivial semi-conformal vectors. Thus, we can also see easily that the twisted Heisenberg-Virasoro vertex operator algebra is a tensor product of two vertex operator algebras. Based on the variety of semi-conformal vectors of the twisted Heisenberg-Virasoro vertex operator algebra, we describe such a class of vertex operator algebras. In general, for a simple CFT-type vertex operator algebra (V,ω), if its variety $\mathrm{Sc}(V,\omega)$ of semi-conformal vectors contains only finite nontrivial elements with the some conditions, then (V,ω) is isomorphic to a twisted Heisenberg-Virasoro vertex operator algebra. Actually, this result shows a characterization of twisted Heisenberg-Virasoro vertex operator algebras.

In further work, we shall understand the properties of some class of vertex operator algebras whose varieties of semi-conformal vectors are finite sets, which will lead to classifying vertex operator algebras by properties of their varieties of semi-conformal vectors from a geometric viewpoint.

Notation: $\mathbb{C}$ is the complex number field; $\mathbb{R}$ is the real number field; $\mathbb{Z}$ is the set of all integer numbers; $\mathbb{N}$ is the set of all non-negative integer numbers; $\mathbb{Z}_+$ is the set of all positive integer numbers.

§2. The Vertex Operator Algebra Associated to the Twisted Heisenberg-Virasoro Algebra

In the section, we shall review the vertex operator algebra associated to the twisted Heisenberg-Virasoro algebra. You can refer to the Refs.[3, 4, 6] for more details.

Let $\mathbb{C}[t^{\pm 1}]$ be the ring of Laurent polynomials with the variable t. Denoted the Lie algebra of derivations on $\mathbb{C}[t^{\pm}]$ by $\mathrm{Der}(\mathbb{C}[t^{\pm 1}])$. Let $L_n = -t^{n+1}\frac{d}{dt}, n \in \mathbb{Z}$. Then $\mathrm{Der}(\mathbb{C}[t^{\pm 1}])$ has a basis $\{L_n \mid n \in \mathbb{Z}\}$. Let A be the universal central extension of abelian Lie algebra $\mathbb{C}[t^{\pm}]$ with a basis $\{t^n, C_h \mid \in \mathbb{Z}\}$. The twisted Heisenberg -Virasoro algebra is the universal central extension of the semi-direct product Lie algebra $\mathrm{Der}(\mathbb{C}[t^{\pm 1}]) \ltimes A$, denoted by $\mathcal{HV}$. The Lie algebra $\mathcal{HV}$ has a basis

$$\{L_m, t^n, C_v, C_h, C \mid m, n \in \mathbb{Z}\}.$$

The non-trivial Lie bracket relations are as follows

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C_v; \tag{2.1}$$

$$[L_n, t^m] = -mt^{m+n} - \delta_{m+n,0}(n^2+n)C. \tag{2.2}$$

where $m, n \in \mathbb{Z}$.

For convenience, we write t^n as b_n, $\mathcal{HV}$ has a $\mathbb{Z}$-graded structure

$$\mathcal{HV} = \bigoplus_{n\in\mathbb{Z}} \mathcal{HV}_{(n)},$$

where for $n \neq 0$, $\mathcal{HV}_{(n)} = \mathbb{C}L_n \oplus \mathbb{C}b_n$ for $n \neq 0$ and $\mathcal{HV}_{(0)} = Span_{\mathbb{C}}\{L_0, b_0, C_v, C_h, C\}$. So $\mathcal{HV}$ has a triangle decomposition

$$\mathcal{HV} = \mathcal{HV}_+ \oplus \mathcal{HV}_- \oplus \mathcal{HV}_{(0)},$$

where $\mathcal{HV}_+ = \oplus_{n>0}\mathcal{HV}_{(n)}; \mathcal{HV}_- = \oplus_{n>0}\mathcal{HV}_{(-n)}$.

Let $\mathbb{C}$ be a 1-dimensional $\mathcal{HV}_+ \oplus \mathcal{HV}_{(\prime)}$-module as follows

$$L_n \cdot 1 = 0,\ for\ n > 0; b_n \cdot 1 = 0,\ for\ n > 0; L_0 \cdot 1 = h, b_0 \cdot 1 = h_1,$$

$$C_v \cdot 1 = c_v; C_h \cdot 1 = c_h; C \cdot 1 = c, \quad h, h_1 c_v, c_h, c \in \mathbb{C}.$$

Then we get the induced $\mathcal{HV}$-module

$$M(h, h_1, c_v, c_h, c) = U(\mathcal{HV}) \otimes_{U(\mathcal{HV}_+ \oplus \mathcal{HV}_{(0)})} \mathbb{C} \cong U(\mathcal{HV}_-)(as\ vector\ spaces).$$

$M(h, h_1, c_v, c_h, c)$ is $\mathbb{Z}-$ graded by eigenvalues of the operator

$$L_0 - h \cdot Id : M(h, h_1, c_v, c_h, c) = \bigoplus_{n=0}^{+\infty} M(h, h_1, c_v, c_h, c)_n$$

with $M(h, h_1, c_v, c_h, c)_n = \{v \in M(h, h_1, c_v, c_h, c) | L(0)v = (n+h)v\}$.

Lemma 2.1[2] Let $c_h = 0$, and $c \neq 0$.

(a) If $\frac{h_1}{c} \in \mathbb{C}\backslash\mathbb{Z}$ or $\frac{h_1}{c} = 1$, then the $\mathcal{HV}-$ module $M(h, h_1, c_v, 0, c)$ is irreducible;

(b) If $\frac{h_1}{c} \in \mathbb{Z}\backslash\{1\}$, then $M(h, h_1, c_v, 0, c)$ possesses a singular vector $v \in M(h, h_1, c_v, 0, c)_p$, where $p = \mid \frac{h_1}{c} - 1 \mid$. The factor-module $V = V(h, h_1, c_v, 0, c) = M(h, h_1, c_v, 0, c)/U(\mathcal{HV}_-)v$ is irreducible and its character is $\mathrm{Ch}(V) = (1-q^p)\prod_{j\geq 1}(1-q^j)^{-2}$.

Denoted by $V(c_v, c_h, c) = M(0,0,c_v,c_h,c)$. Denoted by $\mathbf{1} = \mathbf{1}\otimes\mathbf{1}$. Let I be the $\mathcal{HV}$-submodule of $V(c_v,c_h,c)$ generated by $L_{-1}\mathbf{1}$. Then we consider the quotient module $V_{(c_v,c_h,c)} = V(c_v,c_h,c)/I$. And it has a basis

$$\left\{ L_{n_1}L_{n_2}\cdots L_{n_k}b_{m_1}b_{m_2}\cdots b_{m_l}\mathbf{1} \mid \mathbf{k},\mathbf{l}\in\mathbb{N}, \mathbf{n_1}\le\cdots\le\mathbf{n_k}\le-\mathbf{2}; \mathbf{m_1}\le\cdots\mathbf{m_l}\le-\mathbf{1}, \right\}.$$

$V_{(c_v,c_h,c)}$ has a unique maximal proper submodule, so it has an unique irreducible quotient which is denoted by $L_{(c_v,c_h,c)}$. We can define a $\mathbb{N}-$ graded structure on $V_{(c_v,c_h,c)}$ as follows

$$deg(\mathbf{1}) = \mathbf{0};$$

$$deg(L_{-n_1}L_{-n_2}\cdots L_{-n_k}b_{-m_1}b_{-m_2}\cdots b_{-m_l}\mathbf{1}) = \sum_{\mathbf{j=1}}^{\mathbf{k}}\mathbf{n_j} + \sum_{\mathbf{s=1}}^{\mathbf{l}}\mathbf{m_s}.$$

Let $\mathcal{L}(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ and $b(z) = \sum_{n\in\mathbb{Z}} b(n)z^{-n-1}$. Then they satisfy commutation relations as follows

$$[\mathcal{L}(z_1),\mathcal{L}(z_2)] = 2\mathcal{L}(z_2)z_1^{-1}\partial_{z_2}\delta(\frac{z_2}{z_1}) + \partial_{z_2}(\mathcal{L}(z_2))z_1^{-1}\delta(\frac{z_2}{z_1}) + \frac{1}{12}z_1^{-1}\partial_{z_2}^{(3)}\delta(\frac{z_2}{z_1})c_v. \tag{2.3}$$

$$[b(z_1),b(z_2)] = z_1^{-1}\partial_{z_2}\delta(\frac{z_2}{z_1})c_h. \tag{2.4}$$

$$[\mathcal{L}(z_1),b(z_2)] = b(z_2)z_1^{-1}\partial_{z_2}\delta(\frac{z_2}{z_1}) + \partial_{z_2}b(z_2)z_1^{-1}\delta(\frac{z_2}{z_1}) - z_1^{-1}\partial_{z_2}^{(2)}\delta(\frac{z_2}{z_1})c. \tag{2.5}$$

From above the relations, we have the OPE relations

Corollary 2.2 There are the following OPE relations

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{c_v/2}{(z_1-z_2)^4} + \frac{2\mathcal{L}(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}\mathcal{L}(z_2)}{z_1-z_2}; \tag{2.6}$$

$$b(z_1)b(z_2) \sim \frac{c_h}{(z_1-z_2)^2}; \tag{2.7}$$

$$\mathcal{L}(z_1)b(z_2) \sim \frac{\partial_{z_2}b(z_2)}{z_1-z_2} + \frac{b(z_2)}{(z_1-z_2)^2} - \frac{2c}{(z_1-z_2)^3}. \tag{2.8}$$

Theorem 2.3[4] $V_{(c_v,c_h,c)}$ is a $\mathbb{N}-$graded vertex operator algebra with the conformal vector $L_{-2}\mathbf{1}$ and the central charge c_v and are generated strongly by $\{\mathbf{1},\mathcal{L}(\mathbf{z}),\mathbf{b}(\mathbf{z})\}$.

According to the Lemma 2.1 (b) , we have the following results

Corollary 2.4 For $c_h = 0, c\neq 0$, the $\mathcal{HV}$-module $M(0,0,c_h,0,c)$ possesses a singular vector $L_{(-1)}\mathbf{1}$ in $M(0,0,c_v,0,c)_1$. So the factor-module

$$V_{(c_v,0,c)} = M(0,0,c_v,0,c)/U(\mathcal{HV}_-)(L_{(-1)}\mathbf{1})$$

is a simple vertex operator algebra.

Let H_{c_h} be the Heisenberg vertex operator algebra with the level c_h generated by $\{b_n, C_h | n\in\mathbb{Z}\backslash\{0\}\}$. It follows from Proposition 3.1 in Ref. [6] that

Proposition 2.5 If $c_h \neq 0$, the vertex operator algebra $V_{(c_v,c_h,c)}$ is isomorphic to the tensor product $V_{(c'_v,0)} \otimes H_{c_h}$ of a Virasoro vertex operator algebras $V_{(c'_v,0)}$ with the central charge c'_v and H_{c_h}, where $c'_v = c_v - 1 + \frac{12c^2}{c_h}$ and $\omega'' = (\frac{1}{2c_h}(b_{-1})^2 + \frac{c}{c_h}b_{-2})\mathbf{1}$ is the conformal vector of H_{c_h}.

§3. Semi-conformal Vectors of the Vertex Operator Algebra $V_{(c_v,c_h,c)}$

In this section, let $(V, Y, \mathbf{1}, \omega)$ (Abbrev. (V, ω))be a $\mathbb{Z}-$ graded vertex operator algebra(Refs. [15, 23, 28] for details). We shall review basic notions and results associated with semi-conformal vectors for a vertex operator algebra V. This content can be seen in Refs.[14-15]

3.1 First, we review the commutant of a vertex algebra. It's well-known as the coset construction in conformal field theory(Refs.[17-18]).

Definition 3.1[7, 18, 23, 25] Let W be a vertex algebra, and U be any subset of W. The commutant of U in W is defined by

$$C_W(U) = \{v \in W | [Y(u,z), Y(v,w)] = 0, \forall u \in U\} = \{v \in W | u_n v = 0, \forall u \in U, n \geq 0\}.$$

Remark 3.2 Obviously, $\mathbf{1} \in \mathbf{C_W(U)}$. Furthermore, $C_W(U)$ is a vertex subalgebra of W. And we also have $C_W(U) = C_W(< U >)$, where $< U >$ is the vertex subalgebra of W by the subset U.

Remark 3.3 In a VOA (V, ω), let (U, ω') be a subalgebra of V. If $C_V(C_v(U)) = U$, we say $(U, C_V(U))$ forms a Howe pair in V(Refs.[7,25]). According to the conclusions in Refs. [18,23], a subalgebra U can be realized as a commutant subalgebra of V if and only if $(U, C_V(U))$ forms a Howe pair in V.

3.2 For two given vertex algebras (V, Y_V) and (W, Y_W) a homomorphism $f : V \to W$ of vertex algebras satisfies

$$f(Y_V(u,z)v) = Y_W(f(u),z)f(v), \quad \forall u, v \in V; \text{ and } f(1_V) = 1_W. \tag{3.1}$$

If (V, ω_V) and (W, ω_W) are two VOAs with conformal vectors ω_V and ω_W, respectively, then f is said to be conformal if $f(\omega_V) = \omega_W$. We say f is semi-conformal if $f \circ L_V(n) = L_W(n) \circ f$, for all $n \geq -1$. Let (V, ω_V) be a VOA and a vertex subalgebra of (W, ω_W). We say V is a conformal subalgebra (or subVOA) if $\omega_W = \omega_V$, i.e, they have the same conformal vector. If the inclusion from V to W is semi-conformal, then V is called a semi-conformal subalgebra of W and ω_V is called a semi-conformal vector of W.

For a VOA (W, ω_W) with the conformal vector ω_W, let

$$\mathrm{Sc}(W, \omega_W) = \{\omega' | \omega' \text{ is a semi-conformal vector of } (W, \omega_W)\}.$$

Lemma 3.4[8] A vector $\omega' \in W$ is a semi-conformal vector of (W, ω_W) if and only if it satisfies the following conditions

$$\begin{cases} L'(0)\omega' = L(0)\omega' = 2\omega'; \\ L'(1)\omega' = L(1)\omega' = 0; \\ L'(2)\omega' = L(2)\omega' = \frac{c}{2}\mathbf{1}; \\ L'(-1)\omega' = L(-1)\omega'; \\ L'(n)\omega' = L(n)\omega' = 0, n \geq 3. \end{cases}$$

Where $Y(\omega', z) = L'(z) = \sum_{n\in\mathbb{Z}} L'(n)z^{-n-2}, Y(\omega_W, z) = L(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ and $c \in \mathbb{C}$.

Let (W, ω^W) be a general $\mathbb{Z}$-graded vertex operator algebra. The set $\mathrm{Sc}(W, \omega^W)$ forms an affine algebraic variety([8, Theorem 1.1]). In fact, a semi-conformal vector $\omega' \in W$ can be characterized by algebraic equations of degree at most 2 as described in [8, Proposition 2.2]. The algebraic variety $\mathrm{Sc}(W, \omega^W)$ has also a partial order $\preceq$ (See [8, Definition 2.7]), and this partial order can be characterized by algebraic equations in [8, Proposition 2.8].

Proposition 3.5 If $c_h \neq 0$, then $\mathrm{Sc}(V_{(c,c_h,c_v)}, \omega) = \{0, \omega', \omega - \omega', \omega\}$, where $\omega' = \frac{1}{2c_h}b(-1)^2\mathbf{1} + \frac{\mathbf{c}}{\mathbf{c_h}}\mathbf{b(-2)1}$. Moreover, there are two longest partial order chain in $\mathrm{Sc}(V_{(c,c_h,c_v)}, \omega)$ such as follows

$$0 \preceq \omega' \preceq \omega; 0 \preceq \omega - \omega' \preceq \omega.$$

Proof Note that the weight-two subspace of $V_{(c,c_h,c_v)}$ is spanned by $\{\omega = L(-2)\mathbf{1}, b(-1)^2\mathbf{1}, b(-2)\mathbf{1}\}$. Set $\omega' = xb(-1)^2\mathbf{1} + yb(-2)\mathbf{1} + zL(-2)\mathbf{1}$, where $x, y, z \in \mathbb{C}$. According to the Lemma 3.4, we have $\omega' \in \mathrm{Sc}(V_{(c,c_h,c_v)}, \omega)$ if and only if x, y, z satisfy that

$$\begin{cases} 4c_hx^2 + 4x = 2x; \\ 2y + 4xc + 4c_hxy = 2y; \\ y = 2cx; \\ 4x^2c_h^2 - 12y^2c_h + 4c_hx - 24yc + c_v = 2xc_h - 12yc + c_v; \ , \\ 4x^2c_h + 4x = 2x; \\ 4xyc_h + 2y + 4cx = 2y; \\ z = 1. \end{cases}$$

Equivalently,

$$\begin{cases} 4x^2c_h = 2x; \\ 4xyc_h = 2y; \\ 2xc_h - 12yc = 4x^2c_h^2 - 12y^2c_h; \\ 2y = 4cx; \\ z = 0. \end{cases}$$

So we have nontrival solutions:$x = -\frac{1}{2c_h}, y = -\frac{c}{c_h}, z = 1$ and $x = \frac{1}{2c_h}, y = \frac{c}{c_h}, z = 0$, i.e, there are only two nontrival semi-conformal vectors $\omega' = \frac{1}{2c_h}b(-1)^2\mathbf{1} + \frac{c}{c_h}b(-2)\mathbf{1}$ and $\omega - \omega'$.

With respect to the partial order $\preceq$ of [8, Definition 2.7], we have two longest partial order chain in $\mathrm{Sc}(V_{(c,c_h,c_v)}, \omega)$ such as follows

$$0 \preceq \omega' \preceq \omega; 0 \preceq \omega - \omega' \preceq \omega.$$

Remark 3.6 For each $\omega' \in \mathrm{Sc}(W, \omega_W)$, it determines a unique dual pair $(C_W(C_W(<\omega'>)), C_W(<\omega'>))$ as semi-conformal subalgebras of (W, ω_W) in the sense of Howe duality in VOA theory. Let (V, ω_V) be a semi-conformal subalgebra of (W, ω_W). Then (V, ω_V) has a unique maximal conformal extension $(C_W(C_W(V)), \omega_V)$ in (W, ω_W) in the sense that if $(V, \omega_V) \subset (U, \omega_V)$, then $(U, \omega_V) \subset (C_W(C_W(V)), \omega_V)$ (see [23, Corollary 3.11.14]).

Lemma 3.7 Let (V, ω) be a $\mathbb{N}-$ graded vertex operator algebra with $V_0 = \mathbb{C}1$ and the conformal vector ω. If $\omega' \in \mathrm{Sc}(V, \omega)$, then $C_V(<\omega'>) \otimes C_V(C_V(<\omega'>))$ is a conformal subalgbra of V, where $<\omega'>$ is the Virasoro VOA generated by ω' in V.

Proof We know that $L'(n) = 0$ on $C_V(<\omega'>)$ and $L(n) = L'(n)$ on $C_V(C_V(<\omega'>))$ for $n \geq -1$, then $C_V(<\omega'>) \cap C_V(C_V(<\omega'>)) = \mathbb{C}1$. So $C_V(<\omega'>) \otimes C_V(C_V(<\omega'>))$ is a conformal subalgebra of V.

Theorem 3.8 For $c_h \neq 0$, the Heisenberg-Virasoro vertex operator algebra $V_{(c_v, c_h, c)}$ is isomorphic to the tensor product $V_{(c_v', 0)} \otimes H_{c_h}$ of the simple Virasoro VOA $V_{(c_v', 0)}$ and the Heisenberg VOA H_{c_h} with the conformal vector $\omega' = \frac{1}{2c_h} b(-1)^2\mathbf{1} + \frac{c}{c_h} b(-2)\mathbf{1}$, where $c_v' = c_v + \frac{12c^2}{c_h} - 1$.

Proof By Remark 3.6, we note that the maximal semi-conformal subalgebra with the conformal vector ω' is the Heisenberg VOA H_{c_h} in $V_{(c_v, c_h, c)}$, i.e., $C_{V_{(c_v,c_h,c)}}(C_{V_{(c_v,c_h,c)}}(<\omega'>)) \cong H_{c_h}$. By Lemma 3.7, we know that $C_{V_{(c_v,c_h,c)}}(<\omega'>)) \otimes H_{c_h}$ is a subVOA of $V_{(c_v,c_h,c)}$. And since $<\omega - \omega'> \otimes H_{c_h} \subset C_{V_{(c_v,c_h,c)}}(<\omega'>)) \otimes H_{c_h}$, then $C_{V_{(c_v,c_h,c)}}(<\omega'>)) \otimes H_{c_h}$ as a subVOA of $V_{(c_v,c_h,c)}$ has at less two generators $\{\omega - \omega', b(-1)\mathbf{1}\}$, where $b(-1)\mathbf{1}$ generates H_{c_h}. We know $V_{(c_v,c_h,c)}$ is also generated by two vectors $\{b(-1)\mathbf{1}, \omega\}$ and $C_{V_{(c_v,c_h,c)}}(<\omega'>)) \cap H_{c_h} = \mathbb{C}\mathbf{1}$, then $C_{V_{(c_v,c_h,c)}}(<\omega'>)) = <\omega - \omega'> = V_{(c_v',0)}$ and $V_{(c_v,c_h,c)} \cong V_{(c_v',0)} \otimes H_{c_h}$, when $c_h \neq 0, c_v' = c_v + \frac{12c^2}{c_h} - 1$.

Lemma 3.9[23] Let V be a simple vertex operator algebra and U be any vertex operator subalgebra(with the same conformal vector ω), for example, $U = <\omega>$. Then the vertex subalgebra

$$C_V(U) = \mathbb{C}1.$$

In particular,

$$KerL_{-1} = C_V(V) = C_V(<\omega>) = \mathbb{C}1.$$

Lemma 3.10[21] Let $(V', Y', 1', \omega'), (V'', Y'', 1'', \omega'')$ be two vertex operator algebras. Then there are

$$C_{V'\otimes V''}(V' \otimes 1'') = C_{V'}(V') \otimes V''; C_{V'\otimes V''}(1' \otimes V'') = V' \otimes C_{V''}(V''),$$

In particular, if V' is simple vertex operator algebra, then

$$C_{V'\otimes V''}(V' \otimes 1'') = 1' \otimes V''.$$

According to above Lemma 3.10, 3.11, we have

Corollary 3.11 When $c_h \neq 0$ and $c_v' = c_v + \frac{12c^2}{c_h} - 1$, we have $C_{V_{(c_v,c_h,c)}}(H_{c_h}) = V_{(c_v',0)}$ and $C_{V_{(c_v,c_h,c)}}(V_{(c_v',0)}) = C_{V_{(c_v',0)}}(V_{(c_v',0)}) \otimes H_{c_h}$.

Corollary 3.12 1) If $c'_v \neq 1 - \frac{6(p-q)^2}{pq}$ for all coprime integer pairs $p, q \geq 2$ and $c'_v = c_v - 1 + 12\frac{c^2}{c_h}$ for $c_h \neq 0$, the vertex operator algebra $V_{(c_v,c_h,c)}$ is a simple vertex operator algebra;

2) If there exists coprime integers $p, q \geq 2$ such that $c'_v = 1 - \frac{6(p-q)^2}{pq}$ and $c'_v = c_v - 1 + 12\frac{c^2}{c_h}$ for $c_h \neq 0$, the vertex operator algebra $V_{(c_v,c_h,c)}$ has a unique simple quotient $L_{(c_v,c_h,c)} = L_{(c'_v,0)} \otimes H_{c_h}$.

Proof If $c'_v \neq c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$ for all coprime integer pairs $p, q \in \{2, 3, \cdots\}$, then $V_{(c'_v,0)}$ is a simple vertex operator algebra. By Theorem 3.8, we get $V_{(c_v,c_h,c)}$ is a simple vertex operator algebra.

If $c'_v = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$ for some coprime integer pairs $p, q \in \{2, 3, \cdots\}$, then $V_{(c'_v,0)}$ is not a simple vertex operator algebra, but it has a unique simple quotient $L_{(c'_v,0)}$. By Theorem 3.8, we know that $V_{(c_v,c_h,c)}$ is not a simple vertex operator algebra, however, it has a unique simple quotient $L_{(c_v,c_h,c)} = L_{(c'_v,0)} \otimes H_{c_h}$.

§4. The Characterization of Twisted Heisenberg-Virasoro Vertex Operator Algebras

In this section, according to the properties of twisted Heisenberg-Virasoro vertex operator algebras, we characterize this class of vertex operator algebras by semi-conformal vectors.

Let V be a simple $\mathbb{N}-$ graded vertex operator algebra with $V_0 = \mathbb{C}1$. Such V is also called *a simple CFT type* vertex operator algebra(Refs. [10-11]). If V satisfies the further condition that $L(1)V_1 = 0$, it is of strong CFT type. Li has shown (Ref. [24]) that such a vertex operator algebra V has a unique non-degenerate invariant bilinear form $<,>$ up to a multiplication of a nonzero scalar. In particular, the restriction of $<,>$ to V_1 endows V_1 with a non-degenerate symmetric invariant bilinear form $< u, v >= u(1)v$ for $u, v \in V_1$. For $v \in V_n$, the component operator $v(n-1)$ is called the zero mode of v. It is well-known that V_1 forms a Lie algebra with the bracket operation $[u, v] = u(0)v$ for $u, v \in V_1$. For a simple CFT-type vertex operator algebra (V, ω), if the bilinear form on V_1 is nondegenerate, we say (V, ω) is a $non-degenerate\ simple\ CFT\ type\ vertex\ operator\ algebra$. Let (V, ω_V) be a semi-conformal subalgebra of (W, ω_W) and (U, ω_V) be a semi-conformal subalgebra of (W, ω_W). If $V \subset U$, then we say (U, ω_V) is a conformal extension of (V, ω_V) in (W, ω_W).

Lemma 4.1 Let (U, ω_U) and (V, ω_U) be two semi-conformal subalgebras of the VOA (W, ω_W). If (U, ω_U) is a conformal extension of (V, ω_U) in (W, ω_W), then

1)
$$C_W(V) = C_W(U);$$

2)
$$\mathrm{Sc}(C_W(V), \omega_W - \omega_U) = \mathrm{Sc}(C_W(U), \omega_W - \omega_U).$$

Proof Since (U, ω_U) is a conformal extension of (V, ω_U) in (W, ω_W), then $C_W(V)$ is a conformal extension of $C_W(U)$ in (W, ω_W) and they are both semi-conformal subalgebras with the conformal vector $\omega_W - \omega_U$. According to Refs.[12,16], we know that there is a unique maximal conformal extension for a semi-conformal subalgebra (S, ω_S), which is realized as the double commutant $(C_W(C_W(S)))$ of (S, ω_S) in (W, ω_W) in the sense that if $(S, \omega_S) \subset (T, \omega_S)$, then $(T, \omega_S) \subset (C_W(C_W(S)), \omega_S)$. So $C_W(C_W(C_W(V))) = C_W(C_W(C_W(U)))$. Since $C_W(C_W(C_W(S))) = C_W(S)$ for a general subalgebra S of W, then we have $C_W(V) = C_W(U)$;

According to the definition of semi-conformal vectors of W, the assert 2) is obvious.

Lemma 4.2 Let (V, ω) be a $\mathbb{Z}$-graded vertex operator algebra and (U, ω') be a vertex subalgebra of V. Then $\omega' \in \mathrm{Sc}(V, \omega)$ if and only if $\mathrm{Sc}(U, \omega') \subset \mathrm{Sc}(V, \omega)$.

Proof Since $\omega' \in \mathrm{Sc}(V, \omega)$, then (U, ω') is a semi-conformal subalgebra of V. For any $\omega'' \in \mathrm{Sc}(U, \omega')$, we have $L''(n) = L'(n)$ on W for $n \geq -1$, where (W, ω'') is a semi-conformal subalgebra of U. Since $\omega' \in \mathrm{Sc}(V, \omega)$, then we have $L(n) = L'(n)$ on U for $n \geq -1$. So we have $L(n) = L''(n)$ on W for $n \geq -1$. Hence $\omega'' \in \mathrm{Sc}(V, \omega)$.

If $\mathrm{Sc}(U, \omega') \subset \mathrm{Sc}(V, \omega)$, it is obvious that $\omega' \in \mathrm{Sc}(V, \omega)$.

Lemma 4.3[8] Let (V, ω) be a nondegenerate simple CFT type vertex operator algebra generated by V_1. Let (V', ω') and (V'', ω'') be two vertex operator subalgebras with possible different conformal vectors. Assume that $(V, \omega) = (V', \omega') \otimes (V'', \omega'')$ is a tensor product of vertex operator algebras (see [12, Section 3.12]). Then

1) (V', ω') and (V'', ω'') are semi-conformal subalgebras and both are also non-degenerate simple CFT type;
2) $V_1 = V_1' \otimes \mathbf{1}'' \oplus \mathbf{1}' \otimes \mathbf{V_1''}$, is an orthogonal decomposition with respect to the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V_1;
3) $[V_1' \otimes \mathbf{1}'', \mathbf{1}' \otimes \mathbf{V_1''}] = \mathbf{0}$ with the Lie bracket $[\cdot, \cdot]$ on V_1;
4) $\mathrm{Sc}(V', \omega') \otimes \mathbf{1}''$, $\mathbf{1}' \otimes \mathrm{Sc}(\mathbf{V}'', \omega'')$, and $\mathrm{Sc}(V', \omega') \otimes \mathbf{1}'' + \mathbf{1}' \otimes \mathrm{Sc}(\mathbf{V}'', \omega'')$ are subsets of $\mathrm{Sc}(V, \omega)$;
5) For each $\widetilde{\omega}' \in \mathrm{Sc}(V', \omega')$, we have
$$C_V(< \widetilde{\omega}' > \otimes \mathbf{1}'') = \mathbf{C_{V'}}(< \widetilde{\omega}' >) \otimes \mathbf{V}''$$
and
$$C_V(C_V(< \widetilde{\omega}' > \otimes \mathbf{1}'')) = \mathbf{C_{V'}}(\mathbf{C_{V'}}(< \widetilde{\omega}' >)) \otimes \mathbf{1}''.$$

Lemma 4.4 For a simple CFT type VOA (V, ω), if $V = V^1 \otimes V^2$ and (V^1, ω^1) and (V^2, ω^2) are vertex operator subalgebras of V, then

1) $C_V(< \omega^1 >) = C_V(C_V(< \omega^2 >)) = V^2$ and $C_V(< \omega^2 >) = C_V(C_V(< \omega^1 >)) = V^1$;
2) When $\mathrm{Sc}(V, \omega) = \{0, \omega^1, \omega^2, \omega\}$, we have $\mathrm{Sc}(V^1, \omega^1) = \{0, \omega^1\}$ and $\mathrm{Sc}(V^2, \omega^2) = \{0, \omega^2\}$.

Proof First, we note that $\omega = \omega^1 + \omega^2$. Since $L^1(n) = 0$ on V^2 and $L^2(n) = 0$ on V^1, so $L(n) = L^1(n)$ on V^1 and $L(n) = L^2(n)$ on V^2 for $n \in \mathbb{Z}$, that is $\omega^1, \omega^2 \in \mathrm{Sc}(V, \omega)$.

According to Lemma 3.10, we know that $C_V(V^1) = C_V(C_V(V^2)) = V^2$ and $C_V(V^2) = C_V(C_V(V^1)) = V^1$. Since there exists a unique maximal semi-conformal subalgebra of V for each $\omega' \in \mathrm{Sc}(V,\omega)$, which can be realized as the double commutant subalgebra containing ω' as the conformal vector, then we have $C_V(<\omega^1>) = C_V(C_V(<\omega^2>)) = V^2$ and $C_V(<\omega^2>) = C_V(C_V(<\omega^1>)) = V^1$.

When $\mathrm{Sc}(V,\omega) = \{0,\omega^1,\omega^2,\omega\}$, since $V = V^1 \otimes V^2$, then V^1, V^2 are both semi-conformal subalgebras of V. By Lemma 4.2, we know that $\mathrm{Sc}(V^1,\omega^1) = \{0,\omega^1\}$ and $\mathrm{Sc}(V^2,\omega^2) = \{0,\omega^2\}$.

For a CFT- type VOA (V,ω), we know that V_1 forms a Lie algebra with the bracket operation $[u,v] = u(0)v$ for $u, v \in V_1$.

Lemma 4.5 For a non-degenerate CFT-type vertex operator algebra $V = V^1 \otimes V^2$, where (V^1,ω^1) and (V^2,ω^2) are subVOAs of V, if $\mathrm{Sc}(V,\omega) = \{0,\omega^1,\omega^2,\omega\}$, then either $V_1^1 = 0$ or $V_1^2 = 0$.

Proof Since $V = V^1 \otimes V^2$, by Lemma 4.3 1), we have $V_1 = V_1^1 \oplus V_1^2$ and V_1^1 is orthogonal to V_1^2 in V_1. If $V_1^1 \neq 0$ and $V_1^2 \neq 0$, we take $h_1 \in V_1^1, h_2 \in V_1^2$ such that $<h_i, h_i>= 1, <h_1, h_2>= 0$ for $i = 1, 2$, let $W_1 = Span_{\mathbb{C}}\{h_1, h_2\}$. As an abelian Lie algebra, W_1 generates a Heisenberg vertex operator algebra $M_{W_1}(1)$ with the rank 2. According to Ref. [**?**], we know that $\mathrm{Sc}(M_{W_1}(1))$ is a infinite set, and then by Lemma 4.2, we get $\mathrm{Sc}(M_{W_1}(1)) \subset \mathrm{Sc}(V,\omega)$. So there is a contraction with $\mathrm{Sc}(V,\omega) = \{0,\omega^1,\omega^2,\omega\}$. Therefore, either $V_1^1 = 0$ or $V_1^2 = 0$.

Theorem 4.6 Assume that (V,ω) is a simple non-degenerate CFT type vertex operator algebra and be generated strongly by the subspace $V_1 \oplus V_2$, where $V_1 \neq 0$ is an abelian Lie algebra as the weight-one subspace and V_2 is the weight-two subspace with $\dim V_2 = 1$. If $\mathrm{Sc}(V,\omega) = \{0,\omega',\omega'',\omega\}$ and $V = C_V(<\omega'>) \otimes C_V(<\omega''>)$, then (V,ω) is isomorphic to a simple twisted Heisenberg-Virasoro vertex operator algebra.

Proof Assume that $<\omega'>$ and $<\omega''>$ have central charges c', c'' as Virasoro vertex operator algebras, respectively. At first, since $V = C_V(<\omega'>) \otimes C_V(<\omega''>)$, we note that $\omega'' = \omega - \omega'$ and $C_V(<\omega''>) = C_V(C_V(<\omega'>))$. By Lemma 4.3 2), we have $V_1 = C_V(<\omega'>)_1 \oplus C_V(<\omega''>)_1$ and $C_V(<\omega'>)_1$ is orthogonal to $C_V(<\omega''>)_1$ in V_1. By Lemma 4.5, we know that either $C_V(<\omega'>)_1 = 0$ or $C_V(<\omega''>)_1 = 0$. We can assume that $C_V(<\omega'>)_1 = 0$, then $C_V(<\omega''>)_1 = V_1$.

Since V_1 is an abelian Lie algebra, then V_1 generates a simple Heisenberg VOA $M_{V_1}(c')$ in V and $C_V(<\omega''>) = M_{V_1}(c')$,where c' is the central charge of $M_{V_1}(c')$. According to the condition $\mathrm{Sc}(V,\omega) = \{0,\omega',\omega'',\omega\}$ and the results of Ref.[15], we know that $\dim V_1 = 1$. Note that V is simple, then $C_V(<\omega''>)$ and $C_V(<\omega'>)$ are both simple. On the other hand, since $C_V(<\omega'>)_1 = 0$ and $\dim V_2 = 1$, then $C_V(<\omega'>) =<\omega''>$, where $<\omega''>$ is the simple Virasoro VOA with the central charge c''. Finally, according to Theorem 3.8, we obtain that V is isomorphic to the twisted Heisenberg vertex operator algebra $V_{(c''+c', 1-\frac{12c^2}{c'}, c)}$ or $L_{(c''+c', 1-\frac{12c^2}{c'}, c)}$for some $c \in \mathbb{C}$ as two cases in Corollary 3.12.

The twisted Heisenberg-Virasoro vertex operator algebra has two nontrivial semi-conformal vectors and it is also a tensor product of two vertex operator algebras. Such information will lead us to study the classification of VOAs with two nontrivial semi-conformal vectors in further work.

Remark 4.7 According to our present study, we know that some basic simple CFT type vertex operator algebras have no nontrivial semi-conformal vectors as follows

- $M(1)$(Ref.8), which is the Heisenberg vertex operator algebra with the rank 1 generated by $= \mathbb{C}h$;
- $L_{\hat{\mathrm{sl}}_2}(1,0)$, which is the simple affine type vertex operator algebra associated to sl_2(Ref.[9]);
- $L(\ell,0)$, which is the simple Virasoro vertex operator algebra with the central charge $\ell \neq 0$(Ref.[29]);
- $K(sl_2,\ell)$, which is the parafermion vertex operator algebra with the level $\ell \neq 1$(Refs. [12-13]).
- $V_{\sqrt{k}A_1}$, which are a class of lattice vertex operator algebras associated to root lattice of type A_1 for $k \in \{1,3,4,\cdots\}$(Ref.[28]).

It is interesting problem for us that the classification of vertex operator algebras without nontrivial semi-conformal vectors. Moreover, based on Theorem 4.6, we conjecture that for a vertex operator algebra (V,ω) with two nontrivial semi-conformal vectors, it should contain a conformal vertex operator subalgebra which is a tensor product of two vertex operator algebras without nontrivial semi-conformal vectors up to isomorphism. In fact, we expect to classify vertex operator algebras with two nontrivial semi-conformal vectors by tensor decompositions of vertex operator algebras.

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