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Simple Bounded Weight Modules for the Vector Field Lie Algebras of Infinite Rank

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Abstract: In this paper, we classify the simple uniformly bounded weight modules for the vector field Lie algebra W_∞ of infinite rank. It turns out that any such modules are intermediate series modules. This result is very different from the vector field Lie algebra W_d of finite rank.

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§1. Introduction

Among the theory of infinite dimensional Lie algebras, Lie algebras of polynomial vector fields on irreducible affine algebraic varieties is an important class of Lie algebras. This kind of Lie algebras can describe the symmetries of systems having an infinite number of independent degrees of freedom, for example in Conformal Field Theory, see [2]. Unlike the finite dimensional simple Lie algebras, the representation theory of vector fields Lie algebras at large is still not well developed.

The Lie algebra W_1 of vector fields on a circle is the centerless Virasoso algebra which plays an important role in the quantum field theory. The theory of weight representations of W_1 is well developed, we can refer to the recent monograph [7] for a detailed survey. In [11], Mathieu showed that any simple weight W_1-modules with all finite dimensional weight spaces is a highest/lowest weight module or an intermediate series module. In [12], it was further shown that if a simple weight W_1-module has a nonzero finite dimensional weight space, then all its weight spaces are finite dimensional.

For $d>1$, the representation theory of Lie algebra W_d of vector fields on the d-dimensional torus was studied by [1,5,6,9,10,13] and so on. In [4], Billig and Futorny classified all irreducible Harish-Chandra W_d-modules. In particular, it was shown that irreducible bounded weight W_d-modules can be constructed from simple finite dimensional modules over $\mathfrak{gl}_d$. It is a natural problem to classify simple bounded weight modules for the direct limit Lie algebra $W_\infty=\varinjlim W_d$. In fact, we found that some problems of representation theory of W_∞ is much simpler than

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W_d. We showed that simple uniformly bounded weight modules for W_∞ are intermediate series modules.

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers and complex numbers, respectively. For any Lie algebra $\mathfrak{g}$, we denote its universal enveloping algebra by $U(\mathfrak{g})$.

§2. Preliminaries

In this section we will collect notations and related results.

2.1. Witt algebra W_d and W_∞

For a positive integer d, we denote by W_d the derivation Lie algebra of the Laurent polynomial algebra $A_d = \mathbb{C}[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$. The elements of A_d are spanned by $x^r = x_1^{r_1} x_2^{r_2} \cdots x_d^{r_d}$ where $r = (r_1, r_2, \cdots, r_d) \in \mathbb{Z}^d$. Set $\partial_i = x_i \frac{\partial}{\partial x_i}$ for $i = 1, 2, \ldots, d$. Then

$$W_d = \mathrm{Der}(A_d) = \mathrm{Span}_{\mathbb{C}}\{x^r \partial_i | i \in \{1, \cdots, d\}, r \in \mathbb{Z}^d\}$$

called the Witt algebra and its Lie bracket is given by

$$[x^r \partial_i, x^s \partial_j] = s_i x^{r+s} \partial_j - r_j x^{r+s} \partial_i.$$

We can see that $\mathfrak{h}_d = \mathrm{Span}_{\mathbb{C}}\{\partial_1, \partial_2, \ldots, \partial_d\}$ is a Cartan subalgebra of W_d.

In what follows we consider a fixed infinite chain of embeddings of simple Lie algebras

$$W_1 \hookrightarrow W_2 \hookrightarrow \cdots \hookrightarrow W_d \hookrightarrow W_{d+1} \hookrightarrow \cdots.$$

We define W_∞ to be the direct limit Lie algebra $W_\infty = \varinjlim W_d$, i.e., $W_\infty = \mathrm{Der}(A_\infty)$, where $A_\infty = \mathbb{C}[x_1^{\pm 1}, \cdots, x_d^{\pm 1}, \cdots]$. Then

$$\mathfrak{h}_\infty = \mathrm{Span}_{\mathbb{C}}\{\partial_1, \partial_2, \ldots, \partial_d, \ldots\}$$

is a Cartan subalgebra of W_∞.

2.2. Weight modules.

For a $d \in \mathbb{Z} \cup \{\infty\}$, we called a W_d-module M a weight module if the action of $\mathfrak{h}_d$ on M is diagonalizable. i.e.

$$M_\lambda = \{m \in M | hm = \lambda(h)m, \forall\ h \in \mathfrak{h}_d\}.$$

We denote the support of a weight module by

$$\mathrm{Supp} M = \{\lambda \in \mathfrak{h}_d^* \,|\, M_\lambda \neq 0\}.$$

A uniformly bounded weight modules M of W_d is a weight module whose all weight multiplicities are bounded, i.e., there exists $N \in \mathbb{N}$ such that $\dim M_\lambda < N$ for all $\lambda \in \mathfrak{h}_d^*$. When the dimensions of all weight spaces of a weight module M are 1, we called the W_d-module M intermediate series modules.

2.3. Tensor field modules for W_d.

Let $\mathfrak{gl}_d$ be the Lie algebra consisting of all $d\times d$ complex matrices. We denote by E_{ij} the matrix with 1 as (i,j)-th and 0 as others. Let $\mathfrak{h}=\mathrm{Span}\{h_i|1\le i\le d-1\}$ where $h_i=E_{ii}-E_{i+1,i+1}$.

Let $\Lambda^+=\{\lambda\in\mathfrak{h}^*|\lambda(h_i)\in\mathbb{Z}_+$, for $i=1,2,\dots,d-1\}$ be the set of dominant weights with respect to $\mathfrak{h}$. For any $\psi\in\Lambda^+$, let $V(\psi)$ be the simple $\mathfrak{sl}_n$-module with highest weight ψ. We make $V(\psi)$ into a $\mathfrak{gl}_n$ module $V(\psi,b)$ by defining the action of the identity matrix I as some $b\in\mathbb{C}$.

It is known that the exterior product $\bigwedge^k(\mathbb{C}^d)=\mathbb{C}^d\wedge\cdots\wedge\mathbb{C}^d$ (k times)is a $\mathfrak{gl}_d$-module by

$$X(v_1\wedge\cdots\wedge v_k)=v_1\wedge\cdots v_{i-1}\wedge Xv_i\cdots\wedge v_k,\forall X\in\mathfrak{gl}_d,$$

and as $\mathfrak{gl}_d$-modules:

$$\bigwedge^k(\mathbb{C}^d)\cong V(\delta_k,k),\forall 1\le k\le d-1,$$

where $\delta_k\in\mathfrak{h}^*$ is the the fundamental weights defined by $\delta_k(h_j)=\delta_{k,j}$ for all $k,j=1,2,\cdots,d-1$.

Denote $F^\alpha(\psi,b)=V(\psi,b)\otimes A_d$ for any $\psi\in\Lambda^+,b\in\mathbb{C}$, and $\alpha\in\mathbb{C}^d$. Then $F^\alpha(\psi,b)$ becomes a W_d-module (see [9,15]) by defining

$$x^r\partial_i(v\otimes x^n)=(n_i+\alpha_i)v\otimes x^{n+r}+\sum_{j=1}^d(r_jE_{ji}v)\otimes x^{n+r}.$$

In 1996, Eswara Rao determined the necessary and sufficient conditions for these modules to be irreducible, see [5].

A simplified proof was given in [3]. In [4], it was showed that any irreducible bounded weight modules over W_d is isomorphic to some irreducible sub-quotient of $F^\alpha(\psi,b)$ for some $\psi\in\Lambda^+$ and $b\in\mathbb{C}$.

§3. Classification of simple bounded weight modules

In this section, we will give the classification of simple bounded weight W_∞-modules.

3.1. Some lemmas

For $\beta=(\beta_1,\beta_2,\dots)\in\mathbb{Z}^\infty$, let

$$U_\beta=\{u\in U(W_\infty)\,|\,[\partial_i,u]=\beta_iu,\ \forall\ i\in\mathbb{N}\}.$$

In particular, U_0 is the centralizer of $\mathfrak{h}_\infty$ in $U(W_\infty)$. We also denote $U(W_d)_0=U(W_d)\cap U_0$.

Lemma 3.1. *If M is a simple weight W_∞-module and $\lambda\in\mathrm{Supp}M$, then M_λ is a simple U_0-module.*

Proof. Since M is a simple weight W_∞-module, we have $U(W_\infty)\cdot v=M$ for any nonzero $v\in M_\lambda$. It is clear that $U(W_\infty)=\bigoplus_{\beta\in\mathbb{Z}^\infty}U_\beta$ with respect to the adjoint action of $\mathfrak{h}$. Hence

$$U(W_\infty)\cdot v=(\bigoplus_{\beta\in\mathbb{Z}^\infty}U_\beta)\cdot v=\bigoplus_{\lambda\in\mathrm{Supp}M}M_\lambda.$$

It shows that $U_\beta\cdot v=M_{\beta+\lambda}$ for every $\beta\in\mathbb{Z}^\infty$. In particular, $U_0\cdot v=M_\lambda$ for any nonzero $v\in M_\lambda$. So M_λ is a simple U_0-module. □

Lemma 3.2. *Let M be a simple weight W_∞-module with all finite-dimensional weight spaces and $\lambda\in\mathrm{Supp}M$. Then there exists a $k\in\mathbb{N}$ such that M_λ is a simple $U(W_d)_0$-module for any $d\geq k$.*

Proof. By Lemma 3.1, M_λ is a simple U_0-module. Let

$$\rho: U_0\to \mathrm{End}_{\mathbb{C}}(M_\lambda)$$

be the representation associated with the U_0-module M_λ. By the density Theorem, ρ is an epimorphism. Since $\mathrm{End}_{\mathbb{C}}(M_\lambda)$ is finite dimensional, there is a $k\in\mathbb{N}$ such that the restriction

$$\rho|_{U(W_k)_0}: U(W_k)_0\to \mathrm{End}_{\mathbb{C}}(M_\lambda)$$

is also an epimorphism. So M_λ is a simple $U(W_k)_0$-module, and hence a simple $U(W_d)_0$-module for any $d\geq k$, since $U(W_k)_0\subset U(W_d)_0$. □

3.2. Simple uniformly bounded modules for W_∞.

Lemma 3.3. *Let L be a simple uniformly bounded W_d-module which is not multiplicity-free, where $d\in\mathbb{N}$ with $d>1$, $\lambda\in\mathrm{Supp}L$. Then $\dim L_\lambda\geq d$.*

Proof. By the result in [4], we know that $L\cong F^\alpha(\psi,b)$, for $(\psi,b)\neq(\delta_k,k)$, or $L\cong W(\alpha,k)$, where

$$W(\alpha,k)=\bigoplus_{n\in\mathbb{Z}^d}(\mathbb{C}(n+\beta)\wedge\mathbb{C}^d\wedge\cdots\wedge\mathbb{C}^d)\otimes x^n$$

is a simple W_d-submodule of $F^\alpha(\delta_k,k)$, $1\leq k\leq d-1$.

So $\dim L_\lambda=\dim V(\psi,b)$ for $(\psi,b)\neq(\delta_k,k)$, or $\dim L_\lambda=\binom{d}{k-1}$. If $(\psi,b)\neq(\delta_k,k)$, then $V(\psi,b)$ is a simple faithful module over $\mathfrak{sl}_d$, and hence $\dim V(\psi,b)\geq d$. Then the proof follows. □

Next we will recall the intermediate series module for W_∞ defined in [16]. For $\alpha\in\mathbb{C}^\infty, b\in\mathbb{C}$, there is a W_∞-module structure on A_∞ such that

$$(x^r\partial_i)\cdot x^\gamma=(\alpha_i+\gamma_i+br_i)x^{\gamma+r},\quad \gamma\in\mathbb{Z}^\infty, i\in\mathbb{N}.$$

We denote this W_∞-module by $V_{\alpha,b}$. Clearly, $V_{\alpha,b}\cong V_{\alpha+\gamma,b}$ for any $\gamma\in\mathbb{Z}^\infty$.

Lemma 3.4.

(a). The W_∞-module $V_{\alpha,b}$ is reducible if and only if $\alpha\in\mathbb{Z}^\infty$ and $b\in\{0,1\}$.

(b). $V_{\alpha,b}\cong V_{\alpha',b'}$ if and only if $\alpha-\alpha'\in\mathbb{Z}^\infty$ and $b=b'$.

If $\alpha\in\mathbb{Z}^\infty$ and $b=0$, then $\mathbb{C}x^{-\alpha}$ is the unique trivial submodule of $V_{\alpha,0}$. If $\alpha\in\mathbb{Z}^\infty$ and $b=1$, then $\oplus_{\gamma\neq-\alpha}\mathbb{C}x^\gamma$ is the unique nontrivial irreducible submodule of $V_{\alpha,1}$. We use $V'_{\alpha,b}$ to denote the nontrivial irreducible sub-quotient of $V_{\alpha,b}$, i.e., $V'_{\alpha,b}=V_{\alpha,b}$ if $\alpha\notin\mathbb{Z}^\infty$ or $b\notin\{0,1\}$. $V'_{\alpha,0}=V_{\alpha,0}/\mathbb{C}x^{-\alpha}$, $V'_{\alpha,1}=\oplus_{\gamma\neq-\alpha}\mathbb{C}x^\gamma$ if $\alpha\in\mathbb{Z}^\infty$ and $b\in\{0,1\}$.

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