

A Hovey Triple Arising from a Frobenius Pair

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Abstract: We revisit Auslander-Buchweitz approximation theory and find some relations with cotorsion pairs and model category structures. From the notion of weak-cogenerators, we introduce the concept of Frobenius pair (\mathcal{X}, ω) in a triangulated category \mathcal{T} . We show how to construct from a Frobenius pair (\mathcal{X}, ω) a triangulated model structure on \mathcal{X}^\wedge .

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§1. Introduction

The approximation theory has its origin with the concept of injective envelopes described by Eckmann and Schopf in 1953, and it has had a wide development in the context of module categories since the fifties (see, for example [3, 4, 6]). Inspired by the ideas of injective envelopes and projective covers, Auslander and Buchweitz [2] studied in the maximal Cohen-Macaulay approximations for certain modules. Since the appearance of their work, it has influenced numerous subsequent works of researchers. For example, Hernández and his co-authors [9] developed in an analogous theory of approximations in the sense of Auslander and Buchweitz for triangulated categories. In recent years, a powerful machinery for producing approximations via complete cotorsion pairs has been developed by Eklof, Trlifaj, Enochs, Jenda and Göbel in [6–8].

Recall that a model category structure is a way of formally introducing homotopy theory into a category on an abelian category. Model structures are usually not constructed for their own sake but to understand the localization of a given category. The fundamental theorem in this respect is [10, Theorem 1.2.10], which shows that the localization of the category is equivalent to a sub-quotient of it. Although the category is assumed to be bicomplete, the full strength of the assumption is not used. Many authors have done research on model category structures, (see [12, 13, 15, 18]). Hovey [11] made a general study of Quillen model structures and gave a

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method of constructing model structures on abelian categories. Gillespie [14] defined model structures on exact categories and got Hovey's one-to-one correspondence between exact model structures and complete cotorsion pairs, and later generalized by Yang in [17] for triangulated categories.

The main purpose of this paper is to use Auslander-Buchweitz approximation theory in order to develop the theory of Frobenius pairs in triangulated categories \mathcal{T} , which are concepts we will construct from the notions of generators and cogenerators. We will use Frobenius pairs to construct relative cotorsion pairs, seeking to bring the Hovey correspondence between complete cotorsion pairs and model structures. Specifically, from a certain type of Frobenius pair (\mathcal{X}, ω) in \mathcal{T} , we will obtain model category structures on the thick subcategory $\mathcal{X}^\wedge \subseteq \mathcal{T}$ of objects in \mathcal{T} with finite resolution dimension relative to \mathcal{X} .

This paper is organized as follows.

In Section 2, We recalling some results from Auslander-Buchweitz approximation theory in [9]. We also present in Definition 2.1 the notion of Frobenius pair, which constitutes the main subject studied in this work.

In Section 3, we recall the concept of cotorsion pairs in triangulated categories. In the particular case where $\mathcal{S} \subseteq \mathcal{T}$ is a thick subcategory, a complete cotorsion pair in \mathcal{S} is what we will call in Definition 3.1 an \mathcal{S} -cotorsion pair. Motivated by the interplay between cotorsion pairs and model categories, we show how to obtain from a strong Frobenius pair (\mathcal{X}, ω) two compatible and complete cotorsion pairs in the subcategory $\mathcal{S} = \mathcal{X}^\wedge$, which are examples of relative cotorsion pairs.

In Section 4, We apply the Hovey correspondence to obtain in Theorem 4.2 the main result in this paper: the model structure associated to (\mathcal{X}, ω) , that is, a triangulated model structure on the triangulated subcategory $\mathcal{X}^\wedge \subseteq \mathcal{T}$, whose classes of cofibrant, fibrant and trivial objects are given by \mathcal{X} , \mathcal{X}^\wedge and ω^\wedge , respectively.

§2. Preliminaries

We start this section by collecting all the background material that will be necessary in the sequel. First, we recall the notions of relative projective dimension and resolution dimension of a given class of objects in a triangulated category \mathcal{T} . Later, we also recall definitions and basic properties we need from Auslander-Buchweitz approximation theory. In all that follows, we are taking as the main reference for consulting approximation theory the paper [9] by Hernández and his coauthors.

2.1. Basic definitions

Throughout this paper, \mathcal{T} will be a triangulated category and $[1]: \mathcal{T} \rightarrow \mathcal{T}$ its suspension functor. The term subcategory, in this paper, means a subcategory which is full, additive, and closed under isomorphisms.

Let \mathcal{X} and \mathcal{Y} be classes of objects in \mathcal{T} . We put ${}^\perp \mathcal{X} := \{Z \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(Z, -)|_{\mathcal{X}} = 0\}$ and

$\mathcal{X}^\perp := \{Z \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(-, Z)|_{\mathcal{X}} = 0\}$. We denote by $\mathcal{X} * \mathcal{Y}$ the class of objects $Z \in \mathcal{T}$ for which exists a distinguished triangle $X \rightarrow Z \rightarrow Y \rightarrow X[1]$ in \mathcal{T} with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Furthermore, it is said that \mathcal{X} is *closed under extensions* if $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$.

Recall that a class \mathcal{X} of objects in \mathcal{T} is said to be *suspended* (respectively, *cosuspended*) if $\mathcal{X}[1] \subseteq \mathcal{X}$ (respectively, $\mathcal{X}[-1] \subseteq \mathcal{X}$) and \mathcal{X} is closed under extensions.

Given a class \mathcal{X} of objects in \mathcal{T} , it is said that \mathcal{X} is *closed under cones* if for any distinguished triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ in \mathcal{T} with $A, B \in \mathcal{X}$ we have that $C \in \mathcal{X}$. Similarly, \mathcal{X} is *closed under cocones* if for any distinguished triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ in \mathcal{T} with $B, C \in \mathcal{X}$ we have that $A \in \mathcal{X}$. We denote by $\mathcal{U}_{\mathcal{X}}$ (respectively, ${}_{\mathcal{X}}\mathcal{U}$) the smallest suspended (respectively, cosuspended) subcategory of \mathcal{T} containing the class \mathcal{X} . Note that if \mathcal{X} is suspended (respectively, cosuspended) subcategory of \mathcal{T} , then $\mathcal{X} = \mathcal{U}_{\mathcal{X}}$ (respectively, $\mathcal{X} = {}_{\mathcal{X}}\mathcal{U}$). We also recall that a subcategory \mathcal{A} of \mathcal{T} , which is suspended and cosuspended, is called *triangulated subcategory* of \mathcal{T} . A *thick subcategory* of \mathcal{T} is a triangulated subcategory of \mathcal{T} which is closed under direct summands in \mathcal{T} . We also denote by $\Delta_{\mathcal{T}}(\mathcal{X})$ (respectively, $\overline{\Delta}_{\mathcal{T}}(\mathcal{X})$) to the smallest triangulated (respectively, smallest thick) subcategory of \mathcal{T} containing the class \mathcal{X} .

Resolution and coresolution dimensions Let \mathcal{X} be a class of objects in \mathcal{T} . For any natural number n , we introduce inductively the class $\varepsilon_n^\wedge(\mathcal{X})$ as follows: $\varepsilon_0^\wedge(\mathcal{X}) := \mathcal{X}$ and assuming defined $\varepsilon_{n-1}^\wedge(\mathcal{X})$, the class $\varepsilon_n^\wedge(\mathcal{X})$ is given by all the objects $Z \in \mathcal{T}$ for which there exists a distinguished triangle $Z[-1] \rightarrow W \rightarrow X \rightarrow Z$ in \mathcal{T} with $W \in \varepsilon_{n-1}^\wedge(\mathcal{X})$ and $X \in \mathcal{X}$.

Dually, we set $\varepsilon_0^\vee(\mathcal{X}) := \mathcal{X}$ and supposing defined $\varepsilon_{n-1}^\vee(\mathcal{X})$, the class $\varepsilon_n^\vee(\mathcal{X})$ is formed for all the objects $Z \in \mathcal{T}$ for which there exists a distinguished triangle $Z \rightarrow X \rightarrow K \rightarrow Z[1]$ in \mathcal{T} with $K \in \varepsilon_{n-1}^\vee(\mathcal{X})$ and $X \in \mathcal{X}$.

Let \mathcal{X} be a class of objects in \mathcal{T} and M an object in \mathcal{T} .

(1) $\mathcal{X}^\wedge := \cup_{n \geq 0} \varepsilon_n^\wedge(\mathcal{X})$ and $\mathcal{X}^\vee := \cup_{n \geq 0} \varepsilon_n^\vee(\mathcal{X})$.

(2) The \mathcal{X} -resolution dimension of M is $\text{resdim}_{\mathcal{X}}(M) := \min\{n \in \mathbb{N} \mid M \in \varepsilon_n^\wedge(\mathcal{X})\}$. Dually, the \mathcal{X} -coresolution dimension of M is $\text{coresdim}_{\mathcal{X}}(M) := \min\{n \in \mathbb{N} \mid M \in \varepsilon_n^\vee(\mathcal{X})\}$.

(3) For any subclass \mathcal{Y} of \mathcal{T} , we set $\text{resdim}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\text{resdim}_{\mathcal{X}}(M) \mid M \in \mathcal{Y}\}$. Similarly, we also have $\text{coresdim}_{\mathcal{X}}(\mathcal{Y})$.

Remark 2.1. (1) Observe that a suspended class \mathcal{U} of \mathcal{T} is closed under cones. Similarly, if \mathcal{U} is cosuspended then it is closed under cocones.

(2) Let (\mathcal{Y}, ω) be a pair of classes of objects in \mathcal{T} with $\omega \subseteq \mathcal{Y}$. If \mathcal{Y} is closed under cones (respectively, cocones), then $\omega^\wedge \subseteq \mathcal{Y}$ (respectively, $\omega^\vee \subseteq \mathcal{Y}$).

Relative homological dimensions and weak-cogenerators Let \mathcal{X} be a class of objects in \mathcal{T} and M an object in \mathcal{T} .

(1) The \mathcal{X} -projective dimension of M is

$$\text{pd}_{\mathcal{X}}(M) := \min\{n \in \mathbb{N} \mid \text{Hom}_{\mathcal{T}}(M[-i], -)|_{\mathcal{X}} = 0, \forall i > n\}.$$

(2) The \mathcal{X} -injective dimension of M is

$$\text{id}_{\mathcal{X}}(M) := \min\{n \in \mathbb{N} \mid \text{Hom}_{\mathcal{T}}(-, M[i])|_{\mathcal{X}} = 0, \forall i > n\}.$$

(3) For any class \mathcal{Y} of objects in \mathcal{T} , we set

$$pd_{\mathcal{X}}(\mathcal{Y}) := \sup\{pd_{\mathcal{X}}(C) \mid C \in \mathcal{Y}\} \quad \text{and} \quad id_{\mathcal{X}}(\mathcal{Y}) := \sup\{id_{\mathcal{X}}(C) \mid C \in \mathcal{Y}\}.$$

Let (\mathcal{X}, ω) be a pair of classes of objects in \mathcal{T} . We say that ω is a *weak-cogenerator* in \mathcal{X} , if $\omega \subseteq \mathcal{X} \subseteq \mathcal{X}[-1] * \omega$; ω is a *weak-generator* in \mathcal{X} , if $\omega \subseteq \mathcal{X} \subseteq \omega * \mathcal{X}[1]$; ω is *\mathcal{X} -injective* if $id_{\mathcal{X}}(\omega) = 0$; and dually, ω is *\mathcal{X} -projective* if $pd_{\mathcal{X}}(\omega) = 0$.

Proper class of triangles and ξ -projective A class $\xi \subseteq \Delta$ is called a *proper class of triangles* if the following conditions hold:

- (1) ξ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi \subseteq \Delta$.
- (2) ξ is closed under suspensions and is saturated.
- (3) ξ is closed under base and cobase change.

It is known that Δ_0 and the class of all triangles Δ in \mathcal{T} are proper classes of triangles. There are more interesting example of proper classes of triangles enumerated in [5, Example 2.3]. Throughout the paper, we fix a proper class of triangles ξ in the triangulated category \mathcal{T} .

Recall that an object $P \in \mathcal{T}$ (resp., $I \in \mathcal{T}$) is called *ξ -projective* (resp., *ξ -injective*) if for any triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ in ξ , the induced sequence of abelian groups $0 \rightarrow \text{Hom}_{\mathcal{T}}(P, A) \rightarrow \text{Hom}_{\mathcal{T}}(P, B) \rightarrow \text{Hom}_{\mathcal{T}}(P, C) \rightarrow 0$ (resp., $0 \rightarrow \text{Hom}_{\mathcal{T}}(C, I) \rightarrow \text{Hom}_{\mathcal{T}}(B, I) \rightarrow \text{Hom}_{\mathcal{T}}(A, I) \rightarrow 0$) is exact.

The symbol $\mathcal{P}(\xi)$ (resp. $\mathcal{I}(\xi)$) will denote the full subcategory of ξ -projective (ξ -injective) objects of \mathcal{T} . As an immediate consequence of the above definition we have that the categories $\mathcal{P}(\xi)$ and $\mathcal{I}(\xi)$ are full, additive, closed under isomorphisms, direct summands, and Σ -stable. \mathcal{T} is said to have *enough ξ -projectives* (resp., *ξ -injectives*) if for any objects $X \in \mathcal{T}$ there exists a triangle $K \rightarrow P \rightarrow X \rightarrow K[1]$ in ξ (resp., $X \rightarrow I \rightarrow L \rightarrow X[1]$ in ξ) with $P \in \mathcal{P}(\xi)$ (resp., $I \in \mathcal{I}(\xi)$).

2.2. Fundamental results from Auslander-Buchweitz approximation theory

Keeping in mind the terminology and notation we have presented so far, we are ready to recall the necessary background from Auslander-Buchweitz approximation theory.

Definition 2.1. Let \mathcal{X} and ω be two subcategories of \mathcal{T} . We say that (\mathcal{X}, ω) is a *Frobenius pair* in \mathcal{T} if the following three conditions are satisfied:

- (1) \mathcal{X} is *thick*, that is, $\mathcal{X} = \overline{\Delta}_{\mathcal{T}}(\mathcal{X})$.
- (2) ω is closed under direct summands in \mathcal{T} .
- (3) ω is an \mathcal{X} -injective weak-cogenerator in \mathcal{X} .

If in addition, ω is also an \mathcal{X} -projective weak-generator in \mathcal{X} , then we say that (\mathcal{X}, ω) is a *strong Frobenius pair* in \mathcal{T} .

The proof of the following results can be found in Hernández and his co-authors [9, section 3 and section 5].

Theorem 2.1. [9, Theorem 3.5] Let $\mathcal{X} \subseteq \mathcal{T}$ be a cosuspended subcategory and $\omega \subseteq \mathcal{T}$ be an \mathcal{X} -injective weak-cogenerator in \mathcal{X} . Then, the following conditions hold true:

- (1) $\mathcal{X}^{\wedge} = \Delta_{\mathcal{T}}(\mathcal{X})$.

(2) If \mathcal{X} are closed under direct summands, then $\mathcal{X}^\wedge = \overline{\Delta}_{\mathcal{T}}(\mathcal{X})$.

In particular, If (\mathcal{X}, ω) is a Frobenius pair, then $\mathcal{X}^\wedge = \overline{\Delta}_{\mathcal{T}}(\mathcal{X})$.

Proposition 2.1. [9, Proposition 5.2, 5.3] Let \mathcal{X} and ω be two subcategories of \mathcal{T} such that ω is \mathcal{X} -injective. Then, the following conditions hold true:

(1) ω^\wedge is \mathcal{X} -injective.

(2) If in addition ω is a weak-cogenerator in \mathcal{X} which is closed under direct summands in \mathcal{T} , then the following equalities hold:

$$\begin{aligned}\omega &= \mathcal{X} \cap_{\mathcal{X}} \mathcal{U}^\perp[-1] = \{X \in \mathcal{X} \mid id_{\mathcal{X}}(X) = 0\} = \mathcal{X} \cap \omega^\wedge, \\ \mathcal{X} \cap \omega^\vee &= \{X \in \mathcal{X} \mid id_{\mathcal{X}}(X) < \infty\}.\end{aligned}$$

Furthermore, we have that $id_{\mathcal{X}}(M) = coresdim_{\omega}(M)$, for every $M \in \mathcal{X} \cap \omega^\vee$.

Asadollahi and Salarian defined ξ -Gorenstein projective and ξ -Gorenstein injective objects in [1]. We denote ξ -Gorenstein projective (ξ -Gorenstein injective) objects by $\mathcal{GP}(\xi)$ ($\mathcal{GI}(\xi)$).

Example 2.1. Let \mathcal{T} has enough ξ -projectives. Then the pair $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ satisfies the hypothesis in Proposition 2.1. So part (1) implies that $id_{\mathcal{GP}(\xi)}(\mathcal{P}(\xi)^\wedge) = 0$. Note also that $\mathcal{P}(\xi)^\wedge = \widehat{\mathcal{P}(\xi)}$, the class of objects with finite ξ -projective dimension. By [1, Proposition 3.19], we get the first result in the above proposition. On the other hand, [1, Theorem 3.17] implies the second result.

Theorem 2.2. [9, Theorem 5.4] Let (\mathcal{X}, ω) be a pair of classes of objects in a triangulated category \mathcal{T} such that \mathcal{X} is closed under extensions and ω is a weak-cogenerator in \mathcal{X} . Then, the following statements hold.

(1) For all $C \in \varepsilon_n^\wedge(\mathcal{X})$, there exist two distinguished triangles in \mathcal{T} :

$$\begin{aligned}C[-1] &\longrightarrow Y_C \longrightarrow X_C \xrightarrow{\varphi_C} C \text{ with } X_C \in \mathcal{X} \text{ and } Y_C \in \varepsilon_{n-1}^\wedge(\omega), \\ C &\xrightarrow{\varphi^C} Y^C \longrightarrow X^C \longrightarrow C[1] \text{ with } X^C \in \mathcal{X} \text{ and } Y^C \in \varepsilon_n^\wedge(\omega).\end{aligned}$$

(2) If ω is \mathcal{X} -injective, then φ_C is a right \mathcal{X} -approximation, and φ^C is a left ω^\wedge -approximation.

Corollary 2.1. Let \mathcal{X} and ω be two subcategories of \mathcal{T} such that \mathcal{X} is closed under extensions and direct summands in \mathcal{T} , and ω is a weak-cogenerator in \mathcal{X} . If $resdim_{\mathcal{X}}(C) \leq 1$ and $C \in {}^\perp\omega[1]$, then $C \in \mathcal{X}$.

Proof. For every $C \in {}^\perp\omega[1]$ with $resdim_{\mathcal{X}}(C) \leq 1$, by Theorem 2.2, we have a triangle $C[-1] \rightarrow W \rightarrow X \rightarrow C$ with $W \in \omega$ and $X \in \mathcal{X}$, which is split since $C \in {}^\perp\omega[1]$. Hence, $C \in \mathcal{X}$ since \mathcal{X} is closed under direct summands. \square

Theorem 2.3. [9, Theorem 5.6] Let (\mathcal{X}, ω) be a pair of classes of objects in \mathcal{T} which are closed under direct summands in \mathcal{T} . If \mathcal{X} is closed under extensions and ω is an \mathcal{X} -injective weak-cogenerator in \mathcal{X} , then

$$pd_{\omega^\wedge}(C) = pd_{\omega}(C) = resdim_{\mathcal{X}}(C), \quad \forall C \in \mathcal{X}^\wedge.$$

Example 2.2. Let \mathcal{T} has enough ξ -projectives. By Theorem 2.3, we have that for every object M with finite ξ - \mathcal{G} projective dimension

$$\xi\text{-}\mathcal{G}pd(M) = \text{resdim}_{\mathcal{GP}(\xi)}(M) = pd_{\mathcal{P}(\xi)}(M) = pd_{\widehat{\mathcal{P}(\xi)}}(M).$$

In other words, we have that the following conditions are equivalent:

- (1) $\xi\text{-}\mathcal{G}pd(M) \leq n$.
- (2) $\xi \text{xt}_{\xi}^i(M, Q) = 0$ for every $i > n$ and every Q with finite ξ -projective dimension.
- (3) $\xi \text{xt}_{\xi}^i(M, P) = 0$ for every $i > n$ and every ξ -projective P .

This was proven by Asadollahi and Salarian in [1, Proposition 3.19].

Proposition 2.2. [9, Proposition 5.9] Let (\mathcal{X}, ω) be a pair of classes of objects in \mathcal{T} such that ω is closed under direct summands in \mathcal{T} and \mathcal{X} is cosuspended category in \mathcal{T} . If ω is an \mathcal{X} -injective weak-cogenerator in \mathcal{X} , Then $\omega^{\wedge} = \mathcal{X}^{\wedge} \cap \mathcal{X}^{\perp}[-1]$.

Proposition 2.3. Let (\mathcal{X}, ω) be a Frobenius pair in \mathcal{T} . Then $\mathcal{X}^{\wedge} \cap^{\perp}(\omega[n+1]) = \mathcal{X} = \mathcal{X}^{\wedge} \cap^{\perp}(\omega^{\wedge}[n+1])$ for any $n \geq 0$.

Proof. By Proposition 2.1, we know that $\mathcal{X} \subseteq^{\perp}(\omega[n+1])$ and $\mathcal{X} \subseteq^{\perp}(\omega^{\wedge}[n+1])$, then we have $\mathcal{X} \subseteq \mathcal{X}^{\wedge} \cap^{\perp}(\omega[n+1])$ and $\mathcal{X} \subseteq \mathcal{X}^{\wedge} \cap^{\perp}(\omega^{\wedge}[n+1])$. We assert that $\mathcal{X}^{\wedge} \cap^{\perp}(\omega^{\wedge}[n+1]) \subseteq \mathcal{X}$. Indeed, let $C \in \mathcal{X}^{\wedge} \cap^{\perp}(\omega^{\wedge}[n+1])$. Then, by Theorem 2.2, there exists a distinguished triangle

$$C \rightarrow Y \rightarrow X \rightarrow C[1]$$

where $X \in \mathcal{X} \subseteq^{\perp}(\omega^{\wedge}[n+1])$ and $Y \in \omega^{\wedge}$. Since $C \in^{\perp}(\omega^{\wedge}[n+1])$, it follows that $Y \in^{\perp}(\omega^{\wedge}[n+1])$. Now using Theorem 2.3, we get that $\text{resdim}_{\mathcal{X}}(Y) = pd_{\omega^{\wedge}}(Y) = 0$, and thus $Y \in \mathcal{X}$. Hence, $C \in \mathcal{X}$ since \mathcal{X} is cosuspended. The inclusion $\mathcal{X}^{\wedge} \cap^{\perp}(\omega[n+1]) \subseteq \mathcal{X}$ follows similarly. \square

Theorem 2.4. [9, Theorem 5.16] Let (\mathcal{X}, ω) be a pair of classes of objects in \mathcal{T} , \mathcal{X} be cosuspended and ω be closed under direct summands in \mathcal{T} . If ω is an \mathcal{X} -injective weak-cogenerator in \mathcal{X} , the following statements hold.

- (1) $(\omega^{\wedge})^{\vee} = \{C \in \mathcal{X}^{\wedge} \mid id_{\mathcal{X}}(C) < \infty\}$.
- (2) If \mathcal{X} is closed under direct summands in \mathcal{T} , then $(\omega^{\wedge})^{\vee} = \overline{\Delta}_{\mathcal{T}}(\omega)$.

Theorem 2.5. Let $\mathcal{X} \subseteq \mathcal{T}$ be a thick subcategory of \mathcal{T} , and $\mathcal{Y} \subseteq \mathcal{T}$ be a thick subcategory of \mathcal{T} contained in \mathcal{X}^{\wedge} , such that $\omega := \mathcal{X} \cap \mathcal{Y}$ is an \mathcal{X} -injective weak-cogenerator in \mathcal{X} . Then, the following equalities hold:

$$\mathcal{Y} = \omega^{\wedge} = \mathcal{X}^{\wedge} \cap \mathcal{X}^{\perp}[-1] = \mathcal{X}^{\wedge} \cap \mathcal{X} \mathcal{U}^{\perp}[-1] \subseteq \{C \in \mathcal{X}^{\wedge} \cap \omega^{\perp}[-1] \mid id_{\mathcal{X}}(C) < \infty\}.$$

Moreover, if $\{C \in \mathcal{X}^{\wedge} \cap \omega^{\perp}[-1] \mid id_{\mathcal{X}}(C) \leq 1\}$, then the above include relationship can be changed to an equation.

Proof. We split the proof into several parts.

- By Proposition 2.2, we have the equality $\omega^{\wedge} = \mathcal{X}^{\wedge} \cap \mathcal{X}^{\perp}[-1]$.

- We now show that $\omega^{\wedge} = \mathcal{Y}$. Indeed, since $\omega \subseteq \mathcal{Y}$, By Remark 2.1, we have that $\omega^{\wedge} \subseteq \mathcal{Y}$.

Now let $Y \in \mathcal{Y}$. Knowing that $\mathcal{Y} \subseteq \mathcal{X}^{\wedge}$, we get a distinguished triangle in Theorem 2.2, say

$Y[-1] \rightarrow K \rightarrow X \rightarrow Y$ with $X \in \mathcal{X}$ and $K \in \omega^\wedge \subseteq \mathcal{Y}$. Since \mathcal{Y} is closed under extensions, we have that $X \in \mathcal{X} \cap \mathcal{Y} = \omega$. It follows that $Y \in \omega^\wedge$, that is, $\mathcal{Y} \subseteq \omega^\wedge$. So far, we have proven the equalities $\mathcal{Y} = \omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp[-1]$.

• By [9, Proposition 5.9], we have the equality $\mathcal{X}^\wedge \cap \mathcal{X}^\perp[-1] = \mathcal{X}^\wedge \cap_{\mathcal{X}} \mathcal{U}^\perp[-1]$.

• It is only left to prove the equality $\mathcal{Y} \subseteq \{C \in \mathcal{X}^\wedge \cap \omega^\perp[-1] \mid id_{\mathcal{X}}(C) < \infty\}$. Note that we already have $\mathcal{Y} \subseteq \mathcal{X}^\perp[-1] \subseteq \omega^\perp[-1]$, and by the equality in Theorem 2.4 we know that $\mathcal{Y} = \omega^\wedge \subseteq (\omega^\wedge)^\vee = \{C \in \mathcal{X}^\wedge \mid id_{\mathcal{X}}(C) < \infty\}$. So it follows that the containment $\mathcal{Y} \subseteq \{C \in \mathcal{X}^\wedge \cap \omega^\perp[-1] \mid id_{\mathcal{X}}(C) < \infty\}$ holds.

Now consider an object $M \in \mathcal{X}^\wedge \cap \omega^\perp[-1]$ with $id_{\mathcal{X}}(M) \leq 1$. Given $X \in \mathcal{X}$, there is a distinguished triangle $X \rightarrow W \rightarrow X' \rightarrow X[1]$ with $W \in \omega$ and $X' \in \mathcal{X}$. Then, we have an induced sequence $\text{Hom}_{\mathcal{T}}(W[-1], M) \rightarrow \text{Hom}_{\mathcal{T}}(X[-1], M) \rightarrow \text{Hom}_{\mathcal{T}}(X'[-2], M)$ of abelian groups where $\text{Hom}_{\mathcal{T}}(W[-1], M) = 0$ since $M \in \omega^\perp[-1]$, and $\text{Hom}_{\mathcal{T}}(X'[-2], M) = 0$ since $id_{\mathcal{X}}(M) \leq 1$. It follows that $M \in \mathcal{X}^\wedge \cap_{\mathcal{X}} \mathcal{U}^\perp[-1] = \mathcal{Y}$. \square

§3. Relative cotorsion pair

This section is devoted to present the notion of cotorsion pairs relative to a thick subcategory \mathcal{S} of a triangulated category \mathcal{T} with respect to ξ . We begin this section recalling the concept of cotorsion pairs in triangulated categories, and then we introduce the relative \mathcal{S} -cotorsion pairs as complete cotorsion pairs in $\mathcal{S} \subseteq \mathcal{T}$. Later, we provide a characterization for this concept which, along with the results presented in Section 2, allows us to construct relative cotorsion pairs from Frobenius pairs.

3.1. Cotorsion pairs in triangulated categories

Bare's theory [5, Definition 2.4] Let $T: A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a triangle in ξ . We call $h: C \xrightarrow{h} \Sigma A$ the characteristic class of Δ , and usually we denote it by $chT = h$. Let A, C be two objects of T , and consider the class $\xi^*(C, A)$ of all triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in ξ . we define a relation in $\xi^*(C, A)$ as follows: If $T_i: A \xrightarrow{f_i} B_i \xrightarrow{g_i} C \xrightarrow{h_i} \Sigma A$, $i = 1, 2$, are elements of $\xi^*(C, A)$, then we define $T_1 \sim T_2$ if there exists morphism of triangles: Obviously, α is an isomorphism

$$\begin{array}{ccccccc} T_1 : & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C & \xrightarrow{h_1} \Sigma A \\ & \parallel & & \downarrow \alpha & & \parallel & \parallel \\ T_2 : & A & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C & \xrightarrow{h_2} \Sigma A \end{array}$$

and \sim is an equivalence relation on the class $\xi^*(C, A)$. Using base and cobase change, it is easy to see that we can define (as in the case of the classical Bare's theory in an abelian category) a sum in the class $\xi(C, A) := \xi^*(C, A) / \sim$ in such a way that $\xi(C, A)$ becomes an abelian group and $\xi(-, -): \mathcal{T}^{op} \times \mathcal{T} \rightarrow \mathcal{A}b$ becomes an additive bifunctor. Trivially, we have ch , which is an isomorphism of bifunctors: $\xi(-, -) \rightarrow Ph_{\xi}(-, \Sigma -)$.

Definition 3.1. [17, Definition 2.6] A pair of classes $(\mathcal{A}, \mathcal{B})$ in a triangulated category \mathcal{T} is a cotorsion pair with respect to ξ if the following conditions hold:

- (1) $\xi(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
- (2) $\xi(A, C) = 0$ for all $A \in \mathcal{A}$, then $C \in \mathcal{B}$.
- (3) $\xi(C, B) = 0$ for all $B \in \mathcal{B}$, then $C \in \mathcal{A}$.

For convenience, \mathcal{B} that satisfies (1) and (2) is denoted as $A^{\perp, \xi}$, \mathcal{A} that satisfies (1) and (3) is denoted as ${}^{\perp, \xi}\mathcal{B}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ with respect to ξ is said to have enough ξ -projectives if for any $T \in \mathcal{T}$ there is a triangle $B \rightarrow A \rightarrow T \rightarrow \Sigma B$ in ξ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We say it has enough ξ -injectives if for any $T \in \mathcal{T}$ there is a triangle $T \rightarrow B \rightarrow A \rightarrow \Sigma T$ in ξ , where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If both of these hold, we say that $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair with respect to ξ .

3.2. Cotorsion pairs relative to thick subcategories

From now on, we focus on a special type of complete cotorsion pairs in a thick subcategory \mathcal{S} of a triangulated category \mathcal{T} .

Definition 3.2. Let \mathcal{S} be a thick subcategory of a triangulated category \mathcal{T} , and \mathcal{F} and \mathcal{G} be two subcategories of \mathcal{S} (thought as a triangulated category). We say that $(\mathcal{F}, \mathcal{G})$ is a left \mathcal{S} -cotorsion pair in \mathcal{T} if $\mathcal{F} = {}^{\perp, \xi}\mathcal{G}$ and if for every object $S \in \mathcal{S}$ there exists a distinguished triangle $G \rightarrow F \rightarrow S \rightarrow G[1]$ in ξ with $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Similarly, we have the definitions of right \mathcal{S} -cotorsion pair in \mathcal{T} . Finally, by an \mathcal{S} -cotorsion pair $(\mathcal{F}, \mathcal{G})$ in \mathcal{T} we mean that $(\mathcal{F}, \mathcal{G})$ is both a left and right \mathcal{S} -cotorsion pair in \mathcal{T} .

Proposition 3.1. Let \mathcal{S} , \mathcal{F} and \mathcal{G} be subcategories of \mathcal{T} , where \mathcal{S} is thick. Then, $(\mathcal{F}, \mathcal{G})$ is a left \mathcal{S} -cotorsion pair in \mathcal{T} if, and only if, the following conditions hold true:

- (1) \mathcal{F} and \mathcal{G} are subcategories of \mathcal{S} , and \mathcal{F} is closed under direct summands in \mathcal{T} .
- (2) $\xi(\mathcal{F}, \mathcal{G}) = 0$.
- (3) For every $S \in \mathcal{S}$, there exist a distinguished triangle $G \rightarrow F \rightarrow S \rightarrow G[1]$ in ξ with $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Proof. The "only if" part is clear. For the "if" part, suppose that $\mathcal{F}, \mathcal{G} \subseteq \mathcal{T}$ satisfy conditions (1), (2) and (3). It is clear that $\mathcal{F} \subseteq {}^{\perp, \xi}\mathcal{G}$. Now let $S \in {}^{\perp, \xi}\mathcal{G}$ and a distinguished triangle

$$G \rightarrow F \rightarrow S \rightarrow G[1]$$

as in condition (3). Then, $\xi(\mathcal{F}, \mathcal{G}) = 0$, which implies that the above distinguished triangle splits. It follows that S is a direct summand of $F \in \mathcal{F}$, and so we have $S \in \mathcal{F}$ by condition (1). Hence the inclusion ${}^{\perp, \xi}\mathcal{G} \subseteq \mathcal{F}$ follows. \square

3.3. Relative cotorsion pairs from Frobenius pairs

The characterization of (left and right) \mathcal{S} -cotorsion pairs given in Proposition 3.1 allows us to construct easily cotorsion pairs from Frobenius pairs (Recall Definition 2.1). Later on, we will study correspondences between these two notions.

Theorem 3.1. *If (\mathcal{X}, ω) is a Frobenius pair in \mathcal{T} , then $(\mathcal{X}, \omega^\wedge)$ is an \mathcal{X}^\wedge -cotorsion pair in \mathcal{T} . Moreover, the equalities $\omega = \mathcal{X} \cap \omega^\wedge$ and $\omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp[-1]$ from Propositions 2.1 and 2.2 hold true. In particular, ω^\wedge is a thick subcategory of \mathcal{T} .*

Proof. We check conditions (1), (2) and (3) in Proposition 3.1, along with their dual statements, to show that $(\mathcal{X}, \omega^\wedge)$ is an \mathcal{X}^\wedge -cotorsion pair in \mathcal{T} . By Theorem 2.1, $\mathcal{X}^\wedge \subseteq \mathcal{T}$ is a thick subcategory. On the other hand, we have $id_{\mathcal{X}}(\omega^\wedge) = id_{\mathcal{X}}(\omega) = 0$ by Proposition 2.1, and so condition (2) follows. Furthermore, Proposition 2.2 gives us that $\omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp[-1]$, and hence ω^\wedge is closed under direct summands. Then, the dual of condition (1) follows, while (1) holds since \mathcal{X} is closed under direct summands. Note that condition (3) and its dual hold by Theorem 2.2, and hence we conclude that $(\mathcal{X}, \omega^\wedge)$ is an \mathcal{X}^\wedge -cotorsion pair in \mathcal{T} . \square

Theorem 3.2. *If (\mathcal{X}, ω) is a strong Frobenius pair in \mathcal{T} , Then, the following assertions hold true:*

- (a) $(\omega, \mathcal{X}^\wedge)$ is an \mathcal{X}^\wedge -cotorsion pair in \mathcal{T} .
- (b) $\omega^\wedge = \overline{\Delta}_{\mathcal{T}}(\omega)$.

Proof. For part (a), we check again conditions (1), (2) and (3) in Proposition 3.1, along with their duals statements. Conditions (1) and its dual are straight forward. To show condition (2), it suffices to use dual of [9, Lemma 4.6], [9, Lemma 4.2(b)] and the property $pd_{\mathcal{X}}(\omega) = 0$, so that we have the equalities $id_{\omega}(\mathcal{X}^\wedge) = id_{\omega}(\mathcal{X}) = pd_{\mathcal{X}}(\omega) = 0$. To show the dual of condition (3), for any $Y \in \mathcal{X}^\wedge$, since $(\mathcal{X}, \omega^\wedge)$ is a cotorsion pair, we have a ditinguishen triangle $Y \rightarrow W' \rightarrow X' \rightarrow Y[1]$ with $W' \in \omega^\wedge$ and $X' \in \mathcal{X}$. ω is a weak-generator in \mathcal{X} implies that there exists a ditinguishen triangle $X \rightarrow W \rightarrow X' \rightarrow X[1]$ with $W \in \omega$ and $X \in \mathcal{X}$. Then we have the following commutative diagram by base change:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xlongequal{\quad} & X & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Y & \longrightarrow & M & \longrightarrow & W & \longrightarrow & Y[1] \\
 \parallel & & \downarrow & & \downarrow & & \parallel \\
 Y & \longrightarrow & W' & \longrightarrow & X' & \longrightarrow & Y[1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X'[1] & \xlongequal{\quad} & X'[1] & \longrightarrow & 0
 \end{array}$$

The second column implies that $M \in \mathcal{X}^\wedge$ since $W' \in \omega^\wedge \subseteq \mathcal{X}^\wedge$, $X \in \mathcal{X} \subseteq \mathcal{X}^\wedge$ and \mathcal{X}^\wedge is thick. Then the distinguished triangle $Y \rightarrow M \rightarrow W \rightarrow Y[1]$ is our needed. Finally, for (3), let $Y \in \mathcal{X}^\wedge$. By Theorem 3.1, there exists a distinguished triangle $W_1 \rightarrow X_1 \rightarrow Y \rightarrow W_1[1]$ with $W_1 \in \omega^\wedge$ and $X_1 \in \mathcal{X}$, and a distinguished triangle $X_2 \rightarrow W_2 \rightarrow X_1 \rightarrow X_2[1]$ with $W_2 \in \omega$ and $X_2 \in \mathcal{X}$ since ω is a weak-generator in \mathcal{X} . Then we have the following commutative diagram by base change:

The second row implies that $M \in \mathcal{X}^\wedge$ since $W_1 \in \omega^\wedge \subseteq \mathcal{X}^\wedge$, $X_2 \in \mathcal{X} \subseteq \mathcal{X}^\wedge$ and \mathcal{X}^\wedge is thick. Then The distinguished triangle $M \rightarrow W_2 \rightarrow Y \rightarrow M[1]$ is our needed.

$$\begin{array}{ccccccc}
0 & \longrightarrow & Y[-1] & \xlongequal{\quad} & Y[-1] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_2 & \longrightarrow & M & \longrightarrow & W_1 & \longrightarrow & X_2[1] \\
\parallel & & \downarrow & & \downarrow & & \parallel \\
X_2 & \longrightarrow & W_2 & \longrightarrow & X_1 & \longrightarrow & X_2[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y & \xlongequal{\quad} & Y & \longrightarrow & 0
\end{array}$$

We now focus on showing the equality $\omega^\wedge = \overline{\Delta}_{\mathcal{T}}(\omega)$. Indeed, by the equalities in Theorem 2.4, we have that $\mathcal{X} \cap \overline{\Delta}_{\mathcal{T}}(\omega) = \mathcal{X} \cap (\omega^\wedge)^\vee = \{C \in \mathcal{X} \mid \text{id}_{\mathcal{X}}(C) < \infty\}$. On the one hand, by Proposition 2.1, we have the equality $\mathcal{X} \cap \overline{\Delta}_{\mathcal{T}}(\omega) = \{C \in \mathcal{X} \mid \text{id}_{\mathcal{X}}(C) < \infty\} = \mathcal{X} \cap \omega^\vee$. On the other hand, since ω is an \mathcal{X} -projective weak-generator in \mathcal{X} , we can use the dual version of the equality in Proposition 2.1, that is, $\mathcal{X} \cap \omega^\vee = \omega$. Hence, $\mathcal{X} \cap \overline{\Delta}_{\mathcal{T}}(\omega) = \omega$ holds. Setting $\mathcal{Y} := \overline{\Delta}_{\mathcal{T}}(\omega)$ in Theorem 2.5, it follows that $\omega^\wedge = \overline{\Delta}_{\mathcal{T}}(\omega)$. \square

3.4. Hereditary relative cotorsion pairs

So far we know that the concept of \mathcal{S} -cotorsion pair is a description of the completeness of cotorsion pairs in \mathcal{S} . As many complete cotorsion pairs in the literature are hereditary, we will introduce in this section the corresponding property of "being strongly hereditary" for \mathcal{S} -cotorsion pairs. We begin with the definition of hereditary cotorsion pairs.

Definition 3.3. Let $(\mathcal{F}, \mathcal{G})$ be an \mathcal{S} -cotorsion pair in \mathcal{T} . We say that $(\mathcal{F}, \mathcal{G})$ left strongly hereditary if \mathcal{F} is closed under cocones in \mathcal{T} .

Suppose \mathcal{T} has enough ξ -projectives, if $K \rightarrow P \rightarrow A \rightarrow K[1]$ is distinguished triangle with $P \in \mathcal{P}(\xi)$, then we call the object K a first ξ -syzygy of A . An n th ξ -syzygy of A is defined as usual by induction, we denote by $\Omega^i(A)$ the i th ξ -syzygy of A .

Theorem 3.3. Let $(\mathcal{F}, \mathcal{G})$ be a left strongly hereditary \mathcal{S} -cotorsion pair in \mathcal{T} , and set $\omega := \mathcal{F} \cap \mathcal{G}$. If \mathcal{T} has enough ξ -projectives, then the following statements hold true:

(a) $\mathcal{F} = {}^\perp \mathcal{G}[n+1]$ for any $n \geq 0$.

(b) (\mathcal{F}, ω) is a Frobenius pair in \mathcal{T} .

(c) (ω, \mathcal{G}) is a Frobenius pair in \mathcal{T} .

(d) If $\mathcal{G} \subseteq \mathcal{F}^\wedge$, then $(\mathcal{F}, \mathcal{G})$ is an \mathcal{F}^\wedge -cotorsion pair in \mathcal{T} . Moreover, the following equalities hold: $\mathcal{G} = \omega^\wedge = \mathcal{F}^\wedge \cap \mathcal{F}^\perp[-1]$, $\mathcal{F} = \mathcal{F}^\wedge \cap {}^\perp(\omega[1]) = \mathcal{F}^\wedge \cap {}^\perp(\omega^\wedge[1])$.

Proof. (a) For every $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $n \geq 1$, we have a natural isomorphism $\text{Hom}(F[-n-1], G) = \text{Hom}(\Omega^n(F)[-1], G) = 0$ where $\Omega^n(F) \in \mathcal{F}$ since \mathcal{F} is closed under cocones in \mathcal{T} .

(b) The subcategory $\mathcal{F} = {}^\perp \mathcal{G}[1] \cap \mathcal{S}$ is thick subcategory in \mathcal{T} . On the other hand, note also that ω is closed under direct summands. So it remains to check that ω is an \mathcal{F} -injective weak-cogenerator in \mathcal{T} . (a) implies that ω is \mathcal{F} -injective, since $\text{id}_{\mathcal{F}}(\omega) \leq \text{id}_{\mathcal{F}}(\mathcal{G}) = 0$. Now let $F \in \mathcal{F}$. By the dual of condition (3) in Proposition 3.1, there exists a distinguished triangle

$F \rightarrow W \rightarrow K \rightarrow F[1]$ in \mathcal{T} with $W \in \mathcal{G}$ and $K \in \mathcal{F}$. Since \mathcal{F} is closed under extensions, we obtain that $W \in \mathcal{F} \cap \mathcal{G} =: \omega$, proving that ω is a weak-cogenerator in \mathcal{F} .

(c) The equality $\mathcal{G} = \mathcal{F}^\perp[-1] \cap \mathcal{S}$ implies that \mathcal{G} is thick subcategory in \mathcal{T} . Thus, we have that \mathcal{G} is closed under cones and direct summands in \mathcal{T} . $\omega := \mathcal{F} \cap \mathcal{G}$ implies that ω is thick subcategory in \mathcal{T} . The rest of the proof follows as (b).

(d) We check conditions (1), (2) and (3) in Proposition 3.1, along with their dual statements, to show that $(\mathcal{F}, \mathcal{G})$ is an \mathcal{F}^\wedge -cotorsion pair in \mathcal{T} . Since $\mathcal{F} \subseteq \mathcal{S}$ and \mathcal{S} is closed under cones in \mathcal{T} , By Remark 2.1, we have that $\mathcal{F}^\wedge \subseteq \mathcal{S}$. Note that \mathcal{F} and \mathcal{G} are subcategories of \mathcal{F}^\wedge and $\xi_{\mathcal{S}}(\mathcal{F}, \mathcal{G}) = 0$, so the condition (1) and (2) follows. So every $S \in \mathcal{F}^\wedge$ has two distinguished triangles as in Theorem 2.2. Hence, $(\mathcal{F}, \mathcal{G})$ is an \mathcal{F}^\wedge -cotorsion pair in \mathcal{T} .

To show the equalities, we start noting by (b) and by Theorem 3.1, the equalities $\omega^\wedge = \mathcal{F}^\wedge \cap \mathcal{F}^\perp[-1]$ and $\mathcal{F}^\wedge \cap^\perp(\omega[1]) = \mathcal{F} = \mathcal{F}^\wedge \cap^\perp(\omega^\wedge[1])$, thus having the second equality. We assert that $\mathcal{G} = \omega^\wedge$. Indeed, by Remark 2.1, we get that $\omega^\wedge \subseteq \mathcal{G}$. And in order to prove that $\mathcal{G} \subseteq \omega^\wedge$, it suffices to see that $\mathcal{G} \subseteq \mathcal{F}^\wedge \cap \mathcal{F}^\perp[-1]$, which follows by (a), since $\mathcal{G} \subseteq \mathcal{F}^\wedge$. Hence, the first equality also follows. \square

§4. Model category structures from a Frobenius pair

The concept of model category structure was introduced by [16] in 1967, which is a way of formally introducing homotopy theory into a category on an abelian category. Hovey [11] made a general study of Quillen model structures and gave a method of constructing model structures on abelian categories. Yang [17] defined model structures on triangulated categories and got Hovey's one-to-one correspondence between triangulated model structures and complete cotorsion pairs.

In this section, given a strong Frobenius pair (\mathcal{X}, ω) in \mathcal{T} , we will obtain a triangle model structure on \mathcal{X}^\wedge .

Theorem 4.1. [17, Theorem 3.3, Theorem 4.12] Suppose \mathcal{T} has a triangulated model structure with respect to ξ . Let \mathcal{C} denote the full subcategory of cofibrant objects, \mathcal{F} denote the full subcategory of fibrant objects and \mathcal{W} denote the full subcategory of trivial objects. Then:

- (1) \mathcal{W} is a thick subcategory of \mathcal{T} .
- (2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs with respect to ξ .

Conversely, given classes \mathcal{C} , \mathcal{F} and \mathcal{W} satisfying the two conditions above, there is a unique triangulated model structure on \mathcal{T} with respect to ξ such that \mathcal{W} is the class of trivial objects, \mathcal{C} is the class of cofibrant objects and \mathcal{F} is the class of fibrant objects.

Theorem 4.2. Let (\mathcal{X}, ω) be a strong left Frobenius pair in \mathcal{T} . Then, there exists a unique triangulated model structure on \mathcal{X}^\wedge , referred as the Auslander-Buchweitz model structure associated to (\mathcal{X}, ω) , such that \mathcal{X} is the subcategory of cofibrant objects, \mathcal{X}^\wedge is the subcategory of fibrant objects, and ω^\wedge is the subcategory of trivial objects. We will denote this model structure by $(\mathcal{X}, \omega, \mathcal{X}^\wedge)$.

Proof. Now given a strong Frobenius pair (\mathcal{X}, ω) in a triangulated category \mathcal{T} , the triangulated subcategory $\mathcal{X}^\wedge \subseteq \mathcal{T}$ is thick. By Theorems 3.1 and 3.2, we have two \mathcal{X}^\wedge -cotorsion pairs $(\mathcal{X}, \omega^\wedge)$ and $(\omega, \mathcal{X}^\wedge)$ with $\omega = \mathcal{X} \cap \omega^\wedge$. These are complete cotorsion pairs in the thick subcategory $\mathcal{X}^\wedge \subseteq \mathcal{T}$, thus forming a Hovey triple $(\mathcal{X}, \omega, \mathcal{X}^\wedge)$. \square

§5. Conclusion

From the notion of weak-cogenerators, we introduce the concept of Frobenius pair (\mathcal{X}, ω) in a triangulated category \mathcal{T} , and then, we get an \mathcal{X}^\wedge -cotorsion pair $(\mathcal{X}, \omega^\wedge)$ from it. Finally, We show how to construct from a Frobenius pair (\mathcal{X}, ω) a triangulated model structure on \mathcal{X}^\wedge .

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