Chin. Quart. J. of Math.
2014, 29 (2): 195—202

Probability Inequalities for Extended Negatively Dependent Random Variables and Their Applications

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Abstract: Some probability inequalities are established for extended negatively dependent (END) random variables. The inequalities extend some corresponding ones for negatively associated random variables and negatively orthant dependent random variables. By using these probability inequalities, we further study the complete convergence for END random variables. We also obtain the convergence rate $O(n^{-1/2}\ln^{1/2} n)$ for the strong law of large numbers, which generalizes and improves the corresponding ones for some known results.
Key words: extended negatively dependent sequence; negatively orthant dependent sequence; probability inequality; complete convergence
2000 MR Subject Classification: 60E15, 60F15
CLC number: O211.4 **Document code**: A
Article ID: 1002–0462 (2014) 02–0195–08
DOI:10.13371/j.cnki.chin.q.j.m.2014.02.006

§1. Introduction

Firstly, let us recall the concept of extended negatively dependent random variables.

Definition 1.1[1] We call random variables $\{X_n, n\geq 1\}$ extended negatively dependent (END, in short) if there exists a constant $M>0$ such that both

$$P(X_1>x_1, X_2>x_2, \cdots, X_n>x_n)\leq M\prod_{i=1}^{n}P(X_i>x_i) \tag{1.1}$$

and

$$P(X_1\leq x_1, X_2\leq x_2, \cdots, X_n\leq x_n)\leq M\prod_{i=1}^{n}P(X_i\leq x_i) \tag{1.2}$$

Received date: 2012-06-08
Foundation item: Supported by the Project of the Feature Specialty of China(TS11496); Supported by the Scientific Research Projects of Fuyang Teacher's College(2009FSKJ09)
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hold for each $n \geq 1$ and all real numbers $x_1,\ x_2,\ \cdots,\ x_n$.

If $M = 1$, the random variables are called negatively orthant dependent(NOD, in short). For more details about NOD random variables, one can refer to Joag-Dev and Proschan[2], Want et al[3-4], Sung[5], Wu[6], and so forth. The concept of END sequence was introduced by Liu[1]. Some applications for END sequence have been found. See for example, Liu[1] obtained the precise large deviations for dependent random variables with heavy tails, Liu[7] studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails, Chen et al[8] for obtained the strong law of large numbers for END random variables, Shen[9] presented some probability inequalities for END sequence and gave some applications, Wang and Wang[10] investigated the extended precise large deviations of random sums in the presence of END structure and consistent variation, and so forth. It is easily seen that independent random variables and NOD random variables are END. Joag-Dev and Proschan[2] pointed out that NA random variables are NOD. Thus, NA random variables are END. Since END random variables are much weaker than independent random variables, NA random variables and NOD random variables, studying the limit behavior of END sequence is of interest.

It is well known that the probability inequality plays an important role in various proofs of limit theorems. We consider the following probability inequality. For proof, one can refer to Hoeffding[11].

Theorem A If $X_1,\ X_2,\ \cdots,\ X_n$ are independent and $a_i \leq X_i \leq b_i (i = 1, 2, \cdots, n)$, then for any $t > 0$,

$$P\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} EX_i \geq nt\right) \leq \exp\left\{-\frac{2n^2t^2}{\sum\limits_{i=1}^{n}(b_i - a_i)^2}\right\},\quad n \geq 1. \tag{1.3}$$

Since then the inequality was extended to some cases of dependent sequences, such as negatively associated(NA, in short) sequence, negatively orthant dependent(NOD, in short) sequence, and so forth. The main purpose of the paper is to extend Theorem A for independent sequence to the case of extended negatively dependent(END) sequence, which contains independent sequence, NA sequence and NOD sequence as special cases. By using the Hoeffding-type inequality, we further study the complete convergence and strong law of large numbers for END sequence. We obtain the convergence rate $O(n^{-1/2}\ln^{1/2} n)$ for the strong law of large numbers, which generalizes and improves the corresponding ones of Kim and Kim[12], Nooghabi and Azarnoosh[13], Xing et al[14] and Jabbari et al[15].

The following lemmas will be used to prove the main results of the paper.

Lemma 1.1[7] Let random variables $X_1,\ X_2,\ \cdots,\ X_n$ be END.

(i) If $f_1,\ f_2,\ \cdots,\ f_n$ are all nondecreasing(or nonincreasing) functions, then random variables $f_1(X_1),\ f_2(X_2),\ \cdots,\ f_n(X_n)$ are END.

(ii) For each $n \geq 1$, there exists a constant $M > 0$ such that

$$E\left(\prod_{j=1}^{n} X_j^+\right) \leq M \prod_{j=1}^{n} EX_j^+. \tag{1.4}$$

Lemma 1.2 If X is a random variable such that $a \leq X \leq b$, where a and b are finite real numbers, then for any real number h,

$$Ee^{hX} \leq \frac{b - EX}{b - a} e^{ha} + \frac{EX - a}{b - a} e^{hb}. \tag{1.5}$$

Proof Since the exponential function $\exp(hX)$ is convex, its graph is bounded above on the interval $a \leq X \leq b$ by the straight line which connects its ordinates at $X = a$ and $X = b$. Thus

$$e^{hX} \leq \frac{e^{hb} - e^{ha}}{b - a}(X - a) + e^{ha} = \frac{b - X}{b - a} e^{ha} + \frac{X - a}{b - a} e^{hb},$$

which implies (1.5).

Throughout the paper, let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Denote $S_n \doteq \sum\limits_{i=1}^{n} X_i$ and $B_n^2 \doteq \sum\limits_{i=1}^{n} EX_i^2$ for each $n \geq 1$. M denotes a positive constant which may be different in various places.

§2. Main Results and Their Proofs

Theorem 2.1 Let $\{X_n, n \geq 1\}$ be a sequence of END random variables. If there exist two sequences of real numbers $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ such that $a_i \leq X_i \leq b_i$ for each $i \geq 1$, then for any $\varepsilon > 0$ and each $n \geq 1$, there exists a constant $M > 0$ such that

$$P(S_n - ES_n \geq n\varepsilon) \leq M\exp\left\{-\frac{2n^2\varepsilon^2}{\sum\limits_{i=1}^{n}(b_i - a_i)^2}\right\}, \tag{2.1}$$

$$P(S_n - ES_n \leq -n\varepsilon) \leq M\exp\left\{-\frac{2n^2\varepsilon^2}{\sum\limits_{i=1}^{n}(b_i - a_i)^2}\right\} \tag{2.2}$$

and

$$P(|S_n - ES_n| \geq n\varepsilon) \leq 2M\exp\left\{-\frac{2n^2\varepsilon^2}{\sum\limits_{i=1}^{n}(b_i - a_i)^2}\right\}. \tag{2.3}$$

Proof For any $h > 0$, by Markov's inequality, we can see that

$$P(S_n - ES_n \geq n\varepsilon) \leq Ee^{h(S_n - ES_n - n\varepsilon)}. \tag{2.4}$$

It follows from Lemma 1.1(ii) that there exists a constant $M > 0$ such that

$$Ee^{h(S_n - ES_n - n\varepsilon)} = e^{-hn\varepsilon} E\left(\prod_{i=1}^{n} e^{h(X_i - EX_i)}\right) \leq Me^{-hn\varepsilon} \prod_{i=1}^{n} Ee^{h(X_i - EX_i)}. \tag{2.5}$$

Denote $EX_i = \mu_i$ for each $i \geq 1$. By $a_i \leq X_i \leq b_i$ and Lemma 1.2, we have

$$Ee^{h(X_i - EX_i)} \leq e^{-h\mu_i}\left(\frac{b_i - \mu_i}{b_i - a_i} e^{ha_i} + \frac{\mu_i - a_i}{b_i - a_i} e^{hb_i}\right) \doteq e^{L(h_i)}, \tag{2.6}$$

where

$$L(h_i)=-h_ip_i+\ln(1-p_i+p_ie^{h_i}),\ h_i=h(b_i-a_i),\quad p_i=\frac{\mu_i-a_i}{b_i-a_i}.$$

The first two derivatives of $L(h_i)$ are

$$L^{'}(h_i)=-p_i+\frac{p_i}{(1-p_i)e^{-h_i}+p_i},\ L^{''}(h_i)=\frac{p_i(1-p_i)e^{-h_i}}{[(1-p_i)e^{-h_i}+p_i]^2}. \tag{2.7}$$

The last ratio is of the form $u(1-u)$, where $0<u<1$. Hence

$$L^{''}(h_i)=\frac{(1-p_i)e^{-h_i}}{(1-p_i)e^{-h_i}+p_i}\left(1-\frac{(1-p_i)e^{-h_i}}{(1-p_i)e^{-h_i}+p_i}\right)\leq\frac{1}{4}. \tag{2.8}$$

Therefore, by Taylor's formula and (2.8), we can get

$$L(h_i)\leq L(0)+L^{'}(0)h_i+\frac{1}{8}h_i^2=\frac{1}{8}h_i^2=\frac{1}{8}h^2(b_i-a_i)^2. \tag{2.9}$$

It follows from (2.6) and (2.9) that

$$Ee^{h(X_i-EX_i)}\leq\exp\left\{\frac{1}{8}h^2(b_i-a_i)^2\right\}. \tag{2.10}$$

By (2.4), (2.5) and (2.10), we have

$$P\left(S_n-ES_n\geq n\varepsilon\right)\leq M\exp\left\{-hn\varepsilon+\frac{1}{8}h^2\sum_{i=1}^{n}(b_i-a_i)^2\right\}. \tag{2.11}$$

It is easily seen that the right-hand side of (2.11) has its minimum at $h=\frac{4n\varepsilon}{\sum\limits_{i=1}^{n}(b_i-a_i)^2}$. Inserting this value in (2.11), we can obtain (2.1) immediately. Since $\{-X_n,n\geq1\}$ is a sequence of END random variables, (2.1) implies (2.2). (2.1) and (2.2) yield (2.3). The proof is complete.

Corollary 2.1 Let $\{X_n,n\geq1\}$ be a sequence of END random variables with common distribution function F. Then for any $\varepsilon>0$ and any $x\in\mathbb{R}$, there exists a constant $M>0$ such that

$$P\left(F_n(x)-F(x)\geq\varepsilon\right)\leq M\exp\left\{-2n\varepsilon^2\right\},\quad n\geq1, \tag{2.12}$$

$$P\left(F_n(x)-F(x)\leq-\varepsilon\right)\leq M\exp\left\{-2n\varepsilon^2\right\},\quad n\geq1 \tag{2.13}$$

and

$$P\left(|F_n(x)-F(x)|\geq\varepsilon\right)\leq 2M\exp\left\{-2n\varepsilon^2\right\},\quad n\geq1, \tag{2.14}$$

where $F_n(x)=\frac{1}{n}\sum\limits_{i=1}^{n}I(X_i\leq x)$ and $I(A)$ stands for the indicator function of the set A.

Proof For fixed x, by Lemma 1.1(i), it is easily seen that $\{I(X_n\leq x),n\geq1\}$ is a sequence of END random variables satisfying $0\leq I(X_n\leq x)\leq1$, $n\geq1$ and $E(F_n(x))=F(x)$. Therefore, (2.12)$\sim$(2.14) follow from Theorem 2.1 immediately.

Corollary 2.2 Under the conditions of Corollary 2.1, $F_n(x)\to F(x)$ completely for any $x\in\mathbb{R}$.

Theorem 2.2 Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $|X_i| \leq c < \infty$ for each $i \geq 1$, where c is a positive constant. Then for any $r > \frac{1}{2}$,

$$n^{-r}(S_n - ES_n) \to 0, \quad \text{completely}, \quad n \to \infty. \tag{2.15}$$

Proof For any $\varepsilon > 0$, it follows from Theorem 2.1 that

$$\sum_{n=1}^{\infty} P\left(|S_n - ES_n| \geq n^r \varepsilon\right) \leq 2M \sum_{n=1}^{\infty} \left[\exp\left(-\frac{\varepsilon^2}{2c^2}\right)\right]^{n^{2r-1}} < \infty,$$

which implies (2.15).

Theorem 2.3 Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ for each $n \geq 1$. If there exists a sequence of positive numbers $\{c_n, n \geq 1\}$ such that $|X_i| \leq c_i$ for each $i \geq 1$, then for any $t > 0$ and $n \geq 1$, there exists a constant $M > 0$ such that

$$E\exp\left\{t\sum_{i=1}^{n} X_i\right\} \leq M\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{tc_i} EX_i^2\right\}.$$

Proof It is easy to check that for all $x \in \mathbb{R}$, the following inequality holds

$$e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}.$$

Thus, by $EX_i = 0$ and $|X_i| \leq c_i$ for each $i \geq 1$, we have

$$\begin{aligned} Ee^{tX_i} &\leq 1 + tEX_i + \frac{1}{2}t^2 E\left[X_i^2 e^{t|X_i|}\right] \\ &= 1 + \frac{1}{2}t^2 E\left[X_i^2 e^{t|X_i|}\right] \\ &\leq 1 + \frac{1}{2}t^2 e^{tc_i} EX_i^2 \\ &\leq \exp\left\{\frac{1}{2}t^2 e^{tc_i} EX_i^2\right\} \end{aligned} \tag{2.16}$$

for any $t > 0$. By Lemma 1.1 and (2.16), there exists a constant $M > 0$ such that

$$E\exp\left\{t\sum_{i=1}^{n} X_i\right\} \leq M\prod_{i=1}^{n} Ee^{tX_i} \leq M\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{tc_i} EX_i^2\right\}. \tag{2.17}$$

This completes the proof of the theorem.

Corollary 2.3 Let $\{X_n, n \geq 1\}$ be a sequence of END random variables such that $|X_i| \leq c_i$ for each $i \geq 1$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers. Then for any $t > 0$ and $n \geq 1$, there exists a constant $M > 0$ such that

$$E\exp\left\{t\sum_{i=1}^{n}(X_i - EX_i)\right\} \leq M\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i} EX_i^2\right\}. \tag{2.18}$$

Proof It is easily seen that $\{X_n - EX_n, n \geq 1\}$ is a sequence of END random variables with $E(X_i - EX_i) = 0$ and $|X_i - EX_i| \leq 2c_i$ for each $i \geq 1$. By Theorem 2.3, there exists a constant $M>0$ such that

$$\begin{aligned}&E\exp\left\{t\sum_{i=1}^{n}(X_i-EX_i)\right\}\\ \leq{}& M\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n}e^{2tc_i}E(X_i-EX_i)^2\right\}\\ \leq{}& M\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n}e^{2tc_i}EX_i^2\right\}.\end{aligned}$$

The proof is complete.

Similarly, we can get the following corollary.

Corollary 2.4 Let $\{X_n, n\geq 1\}$ be a sequence of END random variables such that $|X_i|\leq c_n$ for each $1\leq i\leq n$, $n\geq 1$, where $\{c_n, n\geq 1\}$ is a sequence of positive numbers. Then for any $t>0$ and $n\geq 1$, there exists a constant $M>0$ such that

$$E\exp\left\{t\sum_{i=1}^{n}(X_i-EX_i)\right\}\leq M\exp\left\{\frac{t^2}{2}e^{2tc_n}\sum_{i=1}^{n}EX_i^2\right\}. \tag{2.19}$$

Theorem 2.4 Let $\{X_n, n\geq 1\}$ be a sequence of END random variables such that $|X_i|\leq c_n$ for each $1\leq i\leq n$, $n\geq 1$, where $\{c_n, n\geq 1\}$ is a sequence of positive numbers. Then for any $\varepsilon>0$ such that $\varepsilon\leq eB_n^2/(2c_n)$ and $n\geq 1$, there exists a constant $M>0$ such that

$$P\left(\sum_{i=1}^{n}(X_i-EX_i)\geq\varepsilon\right)\leq M\exp\left\{-\frac{\varepsilon^2}{2eB_n^2}\right\}, \tag{2.20}$$

$$P\left(\sum_{i=1}^{n}(X_i-EX_i)\leq-\varepsilon\right)\leq M\exp\left\{-\frac{\varepsilon^2}{2eB_n^2}\right\} \tag{2.21}$$

and

$$P\left(\left|\sum_{i=1}^{n}(X_i-EX_i)\right|\geq\varepsilon\right)\leq 2M\exp\left\{-\frac{\varepsilon^2}{2eB_n^2}\right\}. \tag{2.22}$$

Proof By Markov's inequality and Corollary 2.4, we have that for any $t>0$, there exists a constant $M>0$ such that

$$\begin{aligned}&P\left(\sum_{i=1}^{n}(X_i-EX_i)\geq\varepsilon\right)\\ \leq{}& e^{-t\varepsilon}E\exp\left\{t\sum_{i=1}^{n}(X_i-EX_i)\right\}\\ \leq{}& M\exp\left\{-t\varepsilon+\frac{t^2}{2}e^{2tc_n}B_n^2\right\}.\end{aligned} \tag{2.23}$$

Taking $t=\varepsilon/(eB_n^2)$, and noting that $2tc_n\leq 1$, we can obtain (2.20). By (2.20),

$$P\left(\sum_{i=1}^{n}(X_i-EX_i)\leq-\varepsilon\right)=P\left(\sum_{i=1}^{n}(-X_i-E(-X_i))\geq\varepsilon\right)\leq M\exp\left\{-\frac{\varepsilon^2}{2eB_n^2}\right\}, \tag{2.24}$$

since $\{-X_n, n\geq 1\}$ is a sequence of END random variables. Combining (2.20) with (2.21), we can get (2.22) immediately. This completes the proof of the theorem.

Corollary 2.5 Let $\{X_n, n\geq 1\}$ be a sequence of identically distributed END random variables. Assume that there exists a positive integer n_0 such that $|X_i|\leq c_n$ for each $1\leq i\leq n$, $n\geq n_0$, where $\{c_n, n\geq 1\}$ is a sequence of positive numbers. Then for any $\varepsilon>0$ such that $\varepsilon\leq eEX_1^2/(2c_n)$ and $n\geq n_0$, there exists a constant $M>0$ such that

$$P\left(\sum_{i=1}^{n}(X_i-EX_i)\geq n\varepsilon\right)\leq M\exp\left\{-\frac{n\varepsilon^2}{2eEX_1^2}\right\}, \tag{2.25}$$

$$P\left(\sum_{i=1}^{n}(X_i-EX_i)\leq -n\varepsilon\right)\leq M\exp\left\{-\frac{n\varepsilon^2}{2eEX_1^2}\right\} \tag{2.26}$$

and

$$P\left(\left|\sum_{i=1}^{n}(X_i-EX_i)\right|\geq n\varepsilon\right)\leq 2M\exp\left\{-\frac{n\varepsilon^2}{2eEX_1^2}\right\}. \tag{2.27}$$

Theorem 2.5 Let $\{X_n, n\geq 1\}$ be a sequence of identically distributed END random variables. Assume that there exists a positive integer n_0 such that $|X_i|\leq c_n$ for each $1\leq i\leq n$, $n\geq n_0$, where $\{c_n, n\geq 1\}$ is a sequence of positive numbers satisfying

$$0<c_n\leq\left(\frac{enEX_1^2}{8}\right)^{1/3}. \tag{2.28}$$

Denote $\varepsilon_n=\sqrt{2eEX_1^2c_n/n}$. Then for $n\geq n_0$, there exists a constant $M>0$ such that

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i-EX_i)\right|\geq\varepsilon_n\right)\leq 2Me^{-c_n}. \tag{2.29}$$

Proof It is easy to check that $2\varepsilon_nc_n\leq eEX_1^2$ and $n\varepsilon_n^2/(2eEX_1^2)=c_n$. It follows from Corollary 2.5 that for $n\geq n_0$, there exists a constant $M>0$ such that

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i-EX_i)\right|\geq\varepsilon_n\right)\leq 2M\exp\left\{-\frac{n\varepsilon_n^2}{2eEX_1^2}\right\}=2Me^{-c_n}.$$

The proof is complete.

Taking $c_n=\delta\ln n$ and $\delta>1$ in Theorem 2.5, we can get the following result.

Theorem 2.6 Let $\{X_n, n\geq 1\}$ be a sequence of identically distributed END random variables. Assume that there exists a positive integer n_0 such that $|X_i|\leq\delta\ln n$ for each $1\leq i\leq n$, $n\geq n_0$ and some $\delta>1$. Denote $\varepsilon_n=\sqrt{2\delta eEX_1^2\ln n/n}$. Then

$$\sum_{n=1}^{\infty}P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i-EX_i)\right|\geq\varepsilon_n\right)<\infty. \tag{2.30}$$

Remark 2.1 Borel–Cantelli lemma implies that $\frac{1}{n}\sum\limits_{i=1}^{n}(X_i-EX_i)$ converges almost surely with growth rate $O(n^{-1/2}\ln^{1/2}n)$ under the conditions of Theorem 2.6, which generalizes and improves the corresponding ones of Kim and Kim[12], Nooghabi and Azarnoosh[13], Xing et al[14] and Jabbari et al[15].

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