Chin. Quart. J. of Math.
2014, 29 (3): 325—334

Hochschild Homology of Tame Hecke Algebras

HOU Bo

(*School of Mathematics and Information Science, Henan University, Kaifeng* 475001, *China*)

Abstract: Let A be a tame Hecke algebra of type **A**. A new minimal projective bimodule resolution for A is constructed and the dimensions of all the Hochschild homology groups and cyclic homology groups are calculated explicitly.

Key words: Hochschild homology; Hecke algebra; cyclic homology

2000 MR Subject Classification: 16E40, 16E10, 16G10

CLC number: O14.2 **Document code**: A

Article ID: 1002–0462 (2014) 03–0325–10

DOI:10.13371/j.cnki.chin.q.j.m.2014.03.002

§1. Introduction

Hecke algebras play an important role in combinatorics and representation theory. They arise as deformations of the group algebras of finite Coxeter groups and appear as endomorphism algebras of induced representations of finite or p-adic Chevalley groups. They appear as endomorphism algebras of induced representations of finite or p-adic Chevalley groups and give rise to the Kazhdan-Lusztig polynomials which appear in the expression of the canonical basis in terms of the natural basis of Hecke algebras. The Hecke algebra is also present in the geometry of a semisimple group via the equivariant K-theory of the Steinberg variety. This connection plays an important role in the Springer correspondence and the Langlands classification. So there are many good reasons to study the Hecke algebras and their representations.

The methods and techniques of Homological algebra has become an essential tool in the study of the algebraic structure and representation theory. Using the homological methods study the Hecke algebras has made a lot of results. In [17], the authors use Hochschild cohomology as a tool to explore those deformations of skew group algebras that satisfy an expanded definition of graded Hecke algebra. In [15], Opdam and Solleveld study the homological properties of modules over an affine Hecke algebra and prove a comparison result for higher extensions

Received date: 2013-10-08

Foundation item: Supported by the NNSF of China(11301144)

Biography: HOU Bo(1981-), male, native of Kaifeng, Henan, an associated professor of Henan University, Ph.D., engages in representation theory of algebra.

of tempered modules when passing to the Schwartz algebra, a certain topological completion of the affine Hecke algebra. For the symmetric group S_n, in [6] and [12], the authors show that there are just two Morita types of tame blocks of Hecke algebras $\mathcal{H}_q(S_n)$ of type **A** whenever $q=-1$ and $\mathrm{char}\mathbb{K}\neq 2$. This two tame blocks are represented by the principal blocks of $\mathcal{H}_{-1}(S_5)$ and $\mathcal{H}_{-1}(S_4)$, which are Morita equivalent to $A=\mathbb{K}Q/I$ and $A'=\mathbb{K}Q'/I'$, respectively. Where the quivers and ideals are as follows

$$Q: \quad \begin{tikzcd} 1 \arrow[loop left, "\varepsilon"] \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\bar{\alpha}"] \arrow[loop right, "\bar{\varepsilon}"] \end{tikzcd} \qquad I=\langle \alpha\bar{\varepsilon},\ \varepsilon\alpha,\ \bar{\alpha}\varepsilon,\ \bar{\varepsilon}\bar{\alpha},\ \varepsilon^2-\alpha\bar{\alpha},\ \bar{\varepsilon}^2-\bar{\alpha}\alpha\rangle$$

and

$$Q': \quad \begin{tikzcd} 1 \arrow[loop left, "\varepsilon"] \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\bar{\alpha}"] \end{tikzcd} \qquad I'=\langle \varepsilon\alpha,\ \bar{\alpha}\varepsilon,\ \varepsilon^2-(\alpha\bar{\alpha})^2\rangle.$$

By [14], this two Morita types of tame blocks of Hecke algebras $\mathcal{H}_q(S_n)$ are generalized Brauer tree algebras of same type and so that they are derived equivalence. Moreover, the algebras A and A' are selfinjective special biserial algebras, and A is even symmetric Koszul. In this paper we are interested in Hochschild homology of Hecke algebras of type **A**.

The Hochschild homology and cohomology are subtle variants and derived variants of finite-dimensional algebra and play a fundamental role in representation theory of associative algebra. Let Λ be a finite-dimensional algebra(associative with unity) over a field $\mathbb{K}$. Denote by $\Lambda^e:=\Lambda^{op}\otimes_{\mathbb{K}}\Lambda$ the enveloping algebra of Λ. Then the i-th Hochschild homology of Λ is identified with the $\mathbb{K}$-spaces(see [5])

$$HH_i(\Lambda)=\mathrm{Tor}_i^{\Lambda^e}(\Lambda,\Lambda).$$

Hochschild homology is closely related to the oriented cycle and the global dimension of algebras [1, 9-11]. But in general, it is hard to calculate all the Hochschild homology for a given finite-dimensional algebra. For the Hecke algebras of type **A**, the Hochschild cohomology is calculate in [7] and [16] and the Hochschild cohomology ring is consider in [16] and [18]. In this paper, we calculate the $\mathbb{K}$-dimensions of all the Hochschild homology groups of Hecke algebras of type **A** by constructing a new minimal projective bimodule resolution for $A=\mathbb{K}Q/I$.

The paper is structured as follows. In Section 2, we give a new minimal bimodule projective bimodule resolution of A by a family of bases of Koszul dual of A and in Section 3, we use the closed paths in $\mathbb{K}Q$ to give a presentation of the homology complex of A by the minimal bimodule projective bimodule resolution. Furthermore, the $\mathbb{K}$-dimensions of all the Hochschild homology groups are calculated and whenever $\mathrm{char}\mathbb{K}=0$, the $\mathbb{K}$-dimensions of all the cyclic homology groups are given. Since Hochschild homology is invariant under derived equivalence, our result gives information for arbitrary tame blocks of Hecke algebras of type **A**.

Throughout this paper, we fix $\mathbb{K}$ an algebraically closed field with $\mathrm{char}\mathbb{K}\neq 2$, the algebra $A=\mathbb{K}Q/I$ is the symmetric Koszul algebra defined as above and $\otimes:=\otimes_{\mathbb{K}}$. Denote by e and $\bar{e}$ the trivial path at the vertex 1 and 2. For any path p in Q, we denote by $\mathfrak{o}(p)$ and $\mathfrak{t}(p)$ the trivial paths corresponding to the origin and the terminus of p, respectively.

§2. Minimal Projective Bimodule Resolutions

In this section, we construct a minimal projective bimodule resolution for algebra A by a family of bases of its Koszul dual.

Note that A is quadratic, we first consider the Koszul dual $A^!$ of A. It is easy to see that $A^! \cong \mathbb{K}Q/I^\perp$, where $I^\perp$ is an ideal of $\mathbb{K}Q$ generated by $\{\varepsilon^2+\alpha\bar{\alpha},\ \bar{\varepsilon}^2+\bar{\alpha}\alpha\}$. Since $I^\perp$ is a homogeneous ideal, $A^! \cong \mathbb{K}Q/I^\perp = \bigoplus_{i=0}^{\infty} A_i^!$ is a positive graded algebra by grading on the length of paths. Note that the bar involution given by $e\mapsto \bar{e},\ \bar{e}\mapsto e,\ \alpha\mapsto\bar{\alpha},\ \bar{\alpha}\mapsto\alpha,\ \varepsilon\mapsto\bar{\varepsilon}$ and $\bar{\varepsilon}\mapsto\varepsilon$ induces an isomorphism $^- : A\to A$ and an isomorphism $^- : A^!\to A^!$ and it can be extended to an isomorphism of A^e by $a^{op}\otimes b\mapsto \bar{a}^{op}\otimes\bar{b}$ for all $a,\ b\in A$. Then, $\bar{\bar{a}}=a$ for any a in A or $A^!$.

Denote by $(\alpha\bar{\varepsilon}\bar{\alpha}\varepsilon)^l$ the path given by $\alpha\bar{\varepsilon}\bar{\alpha}\varepsilon$ repeat l times with length $4l$ for any $l\geq 1$, $(\alpha\bar{\varepsilon}\bar{\alpha}\varepsilon)^0=e$ and denote by

$$\theta^i=\begin{cases}(\alpha\bar{\varepsilon}\bar{\alpha}\varepsilon)^l, & \text{if } i=4l,\ l\geq 1;\\ (\alpha\bar{\varepsilon}\bar{\alpha}\varepsilon)^l\alpha, & \text{if } i=4l+1,\ l\geq 0;\\ (\alpha\bar{\varepsilon}\bar{\alpha}\varepsilon)^l\alpha\bar{\varepsilon}, & \text{if } i=4l+2,\ l\geq 0;\\ (\alpha\bar{\varepsilon}\bar{\alpha}\varepsilon)^l\alpha\bar{\varepsilon}\bar{\alpha}, & \text{if } i=4l+3,\ l\geq 0.\end{cases}$$

We have a $\mathbb{K}$-basis of homogeneous space $A_n^!$ as following form

$$F^n=\{f_i^n=\varepsilon^{n-i}\theta^i,\ \bar{f}_i^n=\bar{\varepsilon}^{n-i}\bar{\theta}^i \mid 0\leq i\leq n\}$$

for all $n\geq 0$. Clearly, each element f in F^n is a path.

We now let

$$P_n := \bigoplus_{f\in F^n} A\mathfrak{o}(f)\otimes\mathfrak{t}(f)A.$$

Define $d_1: P_1\to P_0$ by

$$d_1(\mathfrak{o}(f)\otimes\mathfrak{t}(f))=\mathfrak{o}(f)\otimes f-f\otimes\mathfrak{t}(f)$$

for $f\in F^1$. Whenever $n\geq 2$, the differential $d_n: P_n\to P_{n-1}$ is given by

(1) If n is odd,

$$d_n(\mathfrak{o}(f_0^n)\otimes\mathfrak{t}(f_0^n))=\mathfrak{o}(f_0^{n-1})\otimes\varepsilon-\mathfrak{o}(f_1^{n-1})\otimes\bar{\alpha}-\varepsilon\otimes\mathfrak{t}(f_0^{n-1})+\alpha\otimes\mathfrak{t}(\bar{f}_2^{n-1}),$$
$$d_n(\mathfrak{o}(f_1^n)\otimes\mathfrak{t}(f_1^n))=\mathfrak{o}(f_2^{n-1})\otimes\bar{\varepsilon}+\mathfrak{o}(f_0^{n-1})\otimes\alpha-\varepsilon\otimes\mathfrak{t}(f_1^{n-1})-\alpha\otimes\mathfrak{t}(\bar{f}_0^{n-1})$$

and for $i\geq 2$,

$$d_n(\mathfrak{o}(f_i^n)\otimes\mathfrak{t}(f_i^n))=\begin{cases}\mathfrak{o}(f_{i-1}^{n-1})\otimes\varepsilon-\mathfrak{o}(f_{i+1}^{n-1})\otimes\bar{\alpha}-\varepsilon\otimes\mathfrak{t}(f_i^{n-1})+\alpha\otimes\mathfrak{t}(\bar{f}_{i+2}^{n-1}), & \text{if } i=4l,\ l\geq 1;\\ \mathfrak{o}(f_{i+1}^{n-1})\otimes\bar{\varepsilon}+\mathfrak{o}(f_{i-1}^{n-1})\otimes\alpha-\varepsilon\otimes\mathfrak{t}(f_i^{n-1})-\alpha\otimes\mathfrak{t}(\bar{f}_{i-2}^{n-1}), & \text{if } i=4l+1,\ l\geq 1;\\ \mathfrak{o}(f_{i-1}^{n-1})\otimes\bar{\varepsilon}-\mathfrak{o}(f_{i+1}^{n-1})\otimes\alpha-\varepsilon\otimes\mathfrak{t}(f_i^{n-1})+\alpha\otimes\mathfrak{t}(\bar{f}_{i+2}^{n-1}), & \text{if } i=4l+2,\ l\geq 0;\\ \mathfrak{o}(f_{i+1}^{n-1})\otimes\varepsilon+\mathfrak{o}(f_{i-1}^{n-1})\otimes\bar{\alpha}-\varepsilon\otimes\mathfrak{t}(f_i^{n-1})-\alpha\otimes\mathfrak{t}(\bar{f}_{i-2}^{n-1}), & \text{if } i=4l+3,\ l\geq 0.\end{cases}$$

(2) If n is even,

$$d_n(\mathfrak{o}(f_0^n)\otimes\mathfrak{t}(f_0^n))=\mathfrak{o}(f_0^{n-1})\otimes\varepsilon-\mathfrak{o}(f_1^{n-1})\otimes\bar{\alpha}+\varepsilon\otimes\mathfrak{t}(f_0^{n-1})-\alpha\otimes\mathfrak{t}(\bar{f}_1^{n-1})$$

and for $i\geq 1$,

$$d_n(\mathfrak{o}(f_i^n)\otimes\mathfrak{t}(f_i^n))=\begin{cases}\mathfrak{o}(f_{i-1}^{n-1})\otimes\varepsilon-\mathfrak{o}(f_{i+1}^{n-1})\otimes\bar{\alpha}+\varepsilon\otimes\mathfrak{t}(f_i^{n-1})+\alpha\otimes\mathfrak{t}(\bar{f}_{i-2}^{n-1}), & \text{if } i=4l,\ l\geq 1;\\ \mathfrak{o}(f_{i+1}^{n-1})\otimes\bar{\varepsilon}+\mathfrak{o}(f_{i-1}^{n-1})\otimes\alpha+\varepsilon\otimes\mathfrak{t}(f_i^{n-1})-\alpha\otimes\mathfrak{t}(\bar{f}_{i+2}^{n-1}), & \text{if } i=4l+1,\ l\geq 0;\\ \mathfrak{o}(f_{i-1}^{n-1})\otimes\bar{\varepsilon}-\mathfrak{o}(f_{i+1}^{n-1})\otimes\alpha+\varepsilon\otimes\mathfrak{t}(f_i^{n-1})+\alpha\otimes\mathfrak{t}(\bar{f}_{i-2}^{n-1}), & \text{if } i=4l+2,\ l\geq 0;\\ \mathfrak{o}(f_{i+1}^{n-1})\otimes\varepsilon+\mathfrak{o}(f_{i-1}^{n-1})\otimes\bar{\alpha}+\varepsilon\otimes\mathfrak{t}(f_i^{n-1})-\alpha\otimes\mathfrak{t}(\bar{f}_{i+2}^{n-1}), & \text{if } i=4l+3,\ l\geq 0.\end{cases}$$

In addition, we can define $d_n(\mathfrak{o}(\bar{f}_i^n)\otimes\mathfrak{t}(\bar{f}_i^n))$ by the definition above and the bar involution, for all $\bar{f}_i^n\in F^n$. Then, it is easy to check that $\mathbb{P}=(P_n,d_n)$ is a complex. Moreover, we have

Proposition 2.1 The complex $\mathbb{P}=(P_n,d_n)$

$$\cdots\to P_{n+1}\xrightarrow{d_{n+1}}P_n\to\cdots\to P_2\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\xrightarrow{\pi}A\to 0$$

is a minimal projective bimodule resolution of A, where π is the multiplication map.

Proof Firstly, by [3, Themrem 2.10.1], the Yoneda algebra $E(A)$ of A is isomorphic to the Koszul dual $A^!$. Note that for any $n\geq 1$ and $f\in F^n$, there is $f=f'\lambda_{f'}$ for some $f'\in F^{n-1}$ and some arrows $\lambda_{f'}\in Q$. Denote by

$$F_{f_i^n}^{n-1}=\{f'\in F^{n-1}\mid \text{there is an arrows } \lambda_{f'}\in Q \text{ such that} f_i^n=f'\lambda_{f'}\}$$

for each $f_i^n\in F^n$. Then for the maximal semisimple subalgebra $A_0\cong A/\mathrm{rad}A$ of A, we have a minimal projective resolution of A_0 as right A-module as follows

$$\cdots\to P'_{n+1}\xrightarrow{b_{n+1}}P'_n\to\cdots\to P'_2\xrightarrow{b_2}P'_1\xrightarrow{b_1}P'_0\longrightarrow A_0\to 0,$$

where $P'_n=\bigoplus_{f\in F^n}\mathfrak{t}(f)A$, the map $b_n:P'_n\to P'_{n-1}$ is given by

$$\mathfrak{t}(f_i^n)a\mapsto\sum_{f'\in F_{f_i^n}^{n-1}}(-1)^{|\lambda_{f'}^i|}\mathfrak{t}(f')\lambda_{f'}a,\qquad |\lambda_{f'}^i|=\begin{cases}1, & \text{if } \lambda_{f'}=\bar{\alpha},\ i \text{ is even,}\\ & \text{or } \lambda_{f'}=\alpha,\ i \text{ is odd;}\\ 0, & \text{otherwise .}\end{cases}$$

Therefore, by [8, Theorem 2.1], the minimal projective resolution of A_0 induces a minimal projective bimodule resolution of A and the differential d_n is given as above.

§3. Hochschild Homology and Cyclic Homology

In this section we replace the homology complex of A induces by the minimal projective resolution in Proposition 2.1 by a complex of closed paths in $\mathbb{K}Q$ and calculate the $\mathbb{K}$-dimensions of Hochschild homology groups and cyclic homology groups(in case $\mathrm{char}\mathbb{K}=0$) of A.

Let X and Y be the sets of paths in $\mathbb{K}Q$, then one defines

$$X\odot Y=\{(p,q)\in X\times Y\mid \mathfrak{t}(p)=\mathfrak{o}(q) \text{ and } \mathfrak{t}(q)=\mathfrak{o}(p)\}$$

and denote by $\mathbb{K}(X \odot Y)$ the vector space spanned by the elements in $X \odot Y$. We call a pair $(p,\ q)$ is a closed path if $\mathfrak{t}(p) = \mathfrak{o}(q)$ and $\mathfrak{t}(q) = \mathfrak{o}(p)$. Let $\mathcal{B} = \{e,\ \bar{e},\ \alpha,\ \bar{\alpha},\ \varepsilon,\ \bar{\varepsilon},\ \varepsilon^2,\ \bar{\varepsilon}^2\}$ be a $\mathbb{K}$-basis of the algebra A. Consider the set $\mathcal{B} \odot F^n$, we have

$$\mathcal{B} \odot F^0 = \{(e,\ e),\ (\varepsilon,\ e),\ (\varepsilon^2,\ e),\ (\bar{e},\ \bar{e}),\ (\bar{\varepsilon},\ \bar{e}),\ (\bar{\varepsilon}^2,\ \bar{e})\}$$

and when $n \geq 1$,

$$\begin{aligned}\mathcal{B} \odot F^n = &\left\{(e,\ f_i^n),\ (\varepsilon, f_i^n),\ (\varepsilon^2,\ f_i^n),\ (\bar{e}, \bar{f}_i^n),\ (\bar{\varepsilon}, \bar{f}_i^n),\ (\bar{\varepsilon}^2, \bar{f}_i^n) \mid i = 4l \text{ or } 4l+3\right\} \\ &\cup \left\{(\alpha, f_i^n),\ (\bar{\alpha},\ \bar{f}_i^n) \mid i = 4l+1 \text{ or } 4l+2\right\}.\end{aligned}$$

Thus, it is easy to see that

$$|\mathcal{B} \odot F^n| = \begin{cases} 16k + 6 + 2j, & \text{if } n = 4k + j,\ j = 0,\ 1,\ 2; \\ 16k + 16, & \text{if } n = 4k + 3. \end{cases}$$

Applying the functor $A \otimes_{A^e} -$ to the minimal projective bimodule resolution $\mathbb{P} = (P_n,\ d_n)$ in Proposition 2.1, we get a homology complex of algebra A. Now, we use vector spaces $\mathbb{K}(\mathcal{B} \odot F^n)$ to give a presentation of this homology complex.

Lemma 3.1 $A \otimes_{A^e} \mathbb{P} \cong \mathbb{N}$, where the complex $\mathbb{N} = (N_n,\ \tau_n)$, $N_n = \mathbb{K}(\mathcal{B} \odot F^n)$ and differential $\tau_n : N_n \to N_{n-1}$ is given by: for any $(b,\ f_i^n)$ in $\mathbb{K}(\mathcal{B} \odot F^n)$,

(1) If n is odd,

$$\begin{aligned}\tau_n(b,\ f_0^n) &= (\varepsilon b, f_0^{n-1}) - (\bar{\alpha} b, f_1^{n-1}) - (b\varepsilon,\ f_0^{n-1}) + (b\alpha, \bar{f}_2^{n-1}), \\ \tau_n(b, f_1^n) &= (\bar{\varepsilon} b,\ f_2^{n-1}) + (\alpha b, f_0^{n-1}) - (b\varepsilon,\ f_1^{n-1}) - (b\alpha, \bar{f}_0^{n-1})\end{aligned}$$

and for $i \geq 2$,

$$\tau_n(b,\ f_i^n) = \begin{cases} (\varepsilon b,\ f_{i-1}^{n-1}) - (\bar{\alpha} b,\ f_{i+1}^{n-1}) - (b\varepsilon,\ f_i^{n-1}) + (b\alpha,\ \bar{f}_{i+2}^{n-1}), & \text{if } i = 4l,\ l \geq 1; \\ (\bar{\varepsilon} b,\ f_{i+1}^{n-1}) + (\alpha b,\ f_{i-1}^{n-1}) - (b\varepsilon,\ f_i^{n-1}) - (b\alpha,\ \bar{f}_{i-2}^{n-1}), & \text{if } i = 4l+1,\ l \geq 1; \\ (\bar{\varepsilon} b, f_{i-1}^{n-1}) - (\alpha b,\ f_{i+1}^{n-1}) - (b\varepsilon,\ f_i^{n-1}) + (b\alpha,\ \bar{f}_{i+2}^{n-1}), & \text{if } i = 4l+2,\ l \geq 0; \\ (\varepsilon b,\ f_{i+1}^{n-1}) + (\bar{\alpha} b,\ f_{i-1}^{n-1}) - (b\varepsilon,\ f_i^{n-1}) - (b\alpha,\ \bar{f}_{i-2}^{n-1}), & \text{if } i = 4l+3,\ l \geq 0. \end{cases}$$

(2) If n is even,

$$\tau_n(b,\ f_0^n) = (\varepsilon b,\ f_0^{n-1}) - (\bar{\alpha} b,\ f_1^{n-1}) + (b\varepsilon,\ f_0^{n-1}) - (b\alpha,\ \bar{f}_2^{n-1})$$

and for $i \geq 1$,

$$\tau_n(b,\ f_i^n) = \begin{cases} (\varepsilon b, f_{i-1}^{n-1}) - (\bar{\alpha} b,\ f_{i+1}^{n-1}) + (b\varepsilon,\ f_i^{n-1}) + (b\alpha, \bar{f}_{i-2}^{n-1}), & \text{if } i = 4l,\ l \geq 1; \\ (\bar{\varepsilon} b,\ f_{i+1}^{n-1}) + (\alpha b,\ f_{i-1}^{n-1}) + (b\varepsilon,\ f_i^{n-1}) - (b\alpha, \bar{f}_{i+2}^{n-1}), & \text{if } i = 4l+1,\ l \geq 0; \\ (\bar{\varepsilon} b,\ f_{i-1}^{n-1}) - (\alpha b, f_{i+1}^{n-1}) + (b\varepsilon,\ f_i^{n-1}) + (b\alpha,\ \bar{f}_{i-2}^{n-1}), & \text{if } i = 4l+2,\ l \geq 0; \\ (\varepsilon b,\ f_{i+1}^{n-1}) + (\bar{\alpha} b, f_{i-1}^{n-1}) + (b\varepsilon,\ f_i^{n-1}) - (b\alpha,\ \bar{f}_{i+2}^{n-1}), & \text{if } i = 4l+3,\ l \geq 0 \end{cases}$$

and the corresponding formulae for $(b,\ \bar{f}_i^n)$ is induces by $\tau_n(b,\ f_i^n)$ and the bar involution.

Proof Let A_0 be the maximal semisimple subalgebra of A. Then one can check that

$$A \otimes_{A^e} P_n = A \otimes_{A_0^e} \bigoplus_{f \in F^n} (\mathfrak{o}(f) \otimes_{\mathbb{K}} \mathfrak{t}(f)) \cong \bigoplus_{\alpha,\beta \in \{e,\ \bar{e}\}} \alpha A \beta \otimes_{\mathbb{K}} \beta F^n \alpha \cong N_n.$$

Moreover, from the isomorphisms above, we have the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & A \otimes_{A^e} P_n & \xrightarrow{\mathbf{1} \otimes d_n} & A \otimes_{A^e} P_{n-1} & \longrightarrow \cdots \\
& \cong \big\downarrow & & \big\downarrow \cong & \\
\cdots \longrightarrow & \mathbb{K}(\mathcal{B} \odot F^n) & \xrightarrow{\tau_n} & \mathbb{K}(\mathcal{B} \odot F^{n-1}) & \longrightarrow \cdots
\end{array}$$

So the differentials τ_n can be induced by d_n in the minimal projective resolution $\mathbb{P}$.

Thus, by the definition, $HH_n(A) = \mathrm{Ker}\tau_n / \mathrm{Im}\tau_{n+1}$ and so that

$$\begin{aligned}
\dim_{\mathbb{K}} HH_n(A) &= \dim_{\mathbb{K}} \mathrm{Ker}\tau_n - \dim_{\mathbb{K}} \mathrm{Im}\tau_{n+1} \\
&= \dim_{\mathbb{K}} N_n - \dim_{\mathbb{K}} \mathrm{Im}\tau_n - \dim_{\mathbb{K}} \mathrm{Im}\tau_{n+1}.
\end{aligned}$$

Consequently, to calculate the $\mathbb{K}$-dimensions of Hochschild homology groups of A, we only need to determine $\dim_{\mathbb{K}} \mathrm{Im}\tau_n$ for all $n \geq 0$, since $\dim_{\mathbb{K}} N_n = |\mathcal{B} \odot F^n|$. Firstly, we define an order on $\mathcal{B}$ by

$$e \prec \varepsilon \prec \varepsilon^2 \prec \alpha \prec \bar{e} \prec \bar{\varepsilon} \prec \bar{\varepsilon}^2 \prec \bar{\alpha}$$

and define an order on $\mathcal{B} \odot F^n$ by the following relation

$$(b, f_i^n) \prec (b',\ f_{i'}^n) \quad \text{if } b \prec b' \ \text{ or } b = b' \text{ but } i < i',$$

for any $(b,\ f_i^n),\ (b',\ f_{i'}^n) \in \mathcal{B} \odot F^n$. Next, we will give the matrix of τ_n under the ordered bases defined above and show the $\mathbb{K}$-dimension of τ_n by this matrix.

We denote still by τ_n the matrix of the differentials τ_n under the ordered bases above. Firstly, for $n = 1, 2, 3$, direct computations show that

$$\mathrm{rank}\tau_1 = 1;\ \mathrm{rank}\tau_2 = 4;\ \mathrm{rank}\tau_3 = 3$$

and so that

$$HH_0(A) = 5;\ HH_1(A) = 3;\ HH_2(A) = 3.$$

Secondly, for $n \geq 4$, we write

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, one can check that the matrix τ_n has following form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & D & 0 & 0 & 0 & E \\ B & 0 & 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & E & 0 & A & 0 & D \\ C & 0 & 0 & 0 & B & 0 & 0 & 0 \end{pmatrix},$$

where

(1) If $n = 4k$ $(k \geq 1)$, then τ_n is a $16k \times (16k + 6)$ matrix with $2k \times (2k + 1)$ matrices

$$A = \begin{pmatrix} 2 & & & & \\ & A_1 & & & \\ & & \ddots & & \\ & & & A_{k-1} & \\ & & & & A' \end{pmatrix}, \ B = \begin{pmatrix} 0 & & & & \\ B' & B_1 & & & \\ & & \ddots & & \\ & & & B_{k-1} & \\ & & & & B'' \end{pmatrix}, \ C = \begin{pmatrix} C_1 & & & \\ & \ddots & & \\ & & C_k & 0 \end{pmatrix}$$

and $2k \times 2k$ matrices

$$D = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_k \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_k \end{pmatrix},$$

where $A_1 = \cdots = A_{k-1} = U$, $A' = (\,1 \quad 1\,)$, $B' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $B'' = (\,0 \quad 1\,)$, $B_1 = \cdots = B_{k-1} = D_1 = \cdots = D_k = X$ and $C_1 = \cdots = C_k = -E_1 = \cdots = -E_k = Y$;

(2) If $n = 4k + 1$ $(k \geq 1)$, then τ_n is a $(16k + 6) \times (16k + 8)$ matrix with $(2k + 1) \times (2k + 1)$ matrices

$$A = \begin{pmatrix} 0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}, \quad D = \begin{pmatrix} 1 & & & \\ & D_1 & & \\ & & \ddots & \\ & & & D_k \end{pmatrix}, \quad E = \begin{pmatrix} -1 & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_k \end{pmatrix}$$

and $2k \times (2k + 1)$ matrices

$$B = \begin{pmatrix} B_1 & & & \\ & \ddots & & \\ & & B_k & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & & & \\ & \ddots & & \\ & & C_k & 0 \end{pmatrix},$$

where $A_1 = \cdots = A_k = V$, $B_1 = \cdots = B_k = E_1 = \cdots = E_k = -X$ and $C_1 = \cdots = C_k = D_1 = \cdots = D_k = Y$;

(3) If $n = 4k + 2$ $(k \geq 1)$, then τ_n is a $(16k + 8) \times (16k + 10)$ matrix with $(2k + 1) \times (2k + 1)$

matrices

$$A=\begin{pmatrix} 2 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix},\quad B=\begin{pmatrix} 0 & & & \\ B' & B_1 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix},\quad C=\begin{pmatrix} -1 & & & \\ & C_1 & & \\ & & \ddots & \\ & & & C_k \end{pmatrix}$$

and $(2k+1)\times(2k+2)$ matrices

$$D=\begin{pmatrix} D_1 & & & \\ & \ddots & & \\ & & D_k & \\ & & & D' \end{pmatrix},\qquad E=\begin{pmatrix} E_1 & & & \\ & \ddots & & \\ & & E_k & \\ & & & E' \end{pmatrix},$$

where $A_1=\cdots=A_{k-1}=U$, $B'=\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $D'=(0\quad 1)$, $E'=(1\quad 0)$, $B_1=\cdots=B_k=D_1=\cdots=D_k=X$ and $C_1=\cdots=C_k=E_1=\cdots=E_k=-Y$;

(4) If $n=4k+3$ $(k\geq 1)$, then τ_n is a $(16k+10)\times(16k+16)$ matrix with $(2k+1)\times(2k+2)$ matrices

$$A=\begin{pmatrix} 0 & & & & \\ & A_1 & & & \\ & & \ddots & & \\ & & & A_k & 0 \end{pmatrix},\ D=\begin{pmatrix} -1 & & & & \\ & D_1 & & & \\ & & \ddots & & \\ & & & D_k & 0 \end{pmatrix},\ E=\begin{pmatrix} 1 & & & & \\ & E_1 & & & \\ & & \ddots & & \\ & & & E_k & 0 \end{pmatrix}$$

and $(2k+2)\times(2k+2)$ matrices

$$B=\begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_{k+1} \end{pmatrix},\qquad C=\begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_{k+1} \end{pmatrix},$$

where $A_1=\cdots=A_k=V$, $B_1=\cdots=B_k=D_1=\cdots=D_k=-X$ and $C_1=\cdots=C_k=E_1=\cdots=E_k=Y$.

Therefore, we obtain the rank of τ_n, for all $n\geq 4$ as follows

Lemma 3.2 For the differential $\tau_n (n\geq 4)$, we have

$$\dim_{\mathbb{K}}\mathrm{Im}\tau_n=\begin{cases} 7k+2, & \text{if } n=4k,\ \ k\geq 1; \\ 7k+1, & \text{if } n=4k+1,\ \ k\geq 1; \\ 7k+4, & \text{if } n=4k+2,\ \ k\geq 1; \\ 7k+3, & \text{if } n=4k+3,\ k\geq 1. \end{cases}$$

Proof Denote by $G:=\begin{pmatrix} A & 0 \\ B & C \\ 0 & A \\ C & B \end{pmatrix}$ and $H:=\begin{pmatrix} A & D & 0 & E \\ 0 & E & A & D \end{pmatrix}$. Then, it is easy to see that $\dim_{\mathbb{K}}\mathrm{Im}\tau_n=\mathrm{rank}\tau_n=\mathrm{rank}G+\mathrm{rank}H$. By the elementary transformations, we have

$$\mathrm{rank}G=\begin{cases} 3k+2, & \text{if } n=4k+j,\ j=0,1,2,\ k\geq 1; \\ 3k+4, & \text{if } n=4k+3,\ k\geq 1 \end{cases}$$

and

$$\mathrm{rank}H=\begin{cases}4k, & \text{if } n=4k,\ k\geq 1;\\ 4k-1, & \text{if } n=4k+1 \text{ or } 4k+3,\ k\geq 1;\\ 4k+2, & \text{if } n=4k+2,\ k\geq 1.\end{cases}$$

Thus, we get the lemma.

Theorem 3.1 Let A be a tame Hecke algebra of type $\mathbf{A}$. Then for $n\geq 1$,

$$\dim_{\mathbb{K}}HH_n(A)=\begin{cases}2k+3, & \text{if } n=4k+j,\ j=0,\ 1,\ 2,\ k\geq 1;\\ 2k+4, & \text{if } n=4k+3,\ k\geq 1.\end{cases}$$

Proof Note that

$$\dim_{\mathbb{K}}N_n=\begin{cases}16k+6+2j, & \text{if } n=4k+j,\ j=0,1,2;\\ 16k+16, & \text{if } n=4k+3.\end{cases}$$

This theorem following from

$$\dim_{\mathbb{K}}HH_n(A)=\dim_{\mathbb{K}}N_n-\dim_{\mathbb{K}}\mathrm{Im}\tau_n-\dim_{\mathbb{K}}\mathrm{Im}\tau_{n+1}.$$

and Lemma 3.2 directly.

Denote by $HC_n(A)$ the n-th cyclic homology group of A(cf [13]).

Corollary 3.1 If $\mathrm{char}\mathbb{K}=0$, then we have

$$\dim_{\mathbb{K}}HC_n(A)=\begin{cases}k, & \text{if } n=4k+1;\\ k+5, & \text{if } n=4k \text{ or } 4k+2;\\ k+1, & \text{if } n=4k+3.\end{cases}$$

Proof By [13, Theorem 4.1.13], we have

$$\begin{aligned}\dim_{\mathbb{K}}HC_n(A)-\dim_{\mathbb{K}}HC_n(\mathbb{K}^2)=&-(\dim_{\mathbb{K}}HC_{n-1}(A)-\dim_{\mathbb{K}}HC_{n-1}(\mathbb{K}^2))\\ &+(\dim_{\mathbb{K}}HH_n(A)-\dim_{\mathbb{K}}HH_n(\mathbb{K}^2)).\end{aligned}$$

Thus $\dim_{\mathbb{K}}HC_n(A)-\dim_{\mathbb{K}}HC_n(\mathbb{K}^2)=\sum\limits_{i=0}^{n}(-1)^{n-i}(\dim_{\mathbb{K}}HH_i(A)-\dim_{\mathbb{K}}HH_i(\mathbb{K}^2))$.

Moreover, it is well-known that

$$\dim_{\mathbb{K}}HH_i(\mathbb{K}^2)=\begin{cases}2, & \text{if } i=0\\ 0, & \text{if } i\geq 1\end{cases} \quad\text{and}\quad \dim_{\mathbb{K}}HC_i(\mathbb{K}^2)=\begin{cases}2, & \text{if } i \text{ is even};\\ 0, & \text{if } i \text{ is odd}.\end{cases}$$

Thus, by Theorem 3.1, we can obtain this corollary directly.

Remark 3.1 Dieter Happel in [10] asked the following question: if the Hochschild cohomology groups $HH^n(\Lambda)$ of a finite dimensional algebra Λ over a field k vanish for all sufficiently large n, is the global dimension of Λ finite? The paper [2] have given a negative answer by the four dimensional algebra $\mathbb{K}\langle x,\ y\rangle/(x^2,\ xy-qyx,\ y^2)$.

In [9], Han conjectured that the homology of Happel's question would always hold, namely that a finite-dimensional algebra whose higher Hochschild homology groups vanish must be of finite global dimension. It is known that Han's conjecture holds for commutative algebras, monomial algebras [1, 9]. If the characteristic of the ground field is zero, Han's conjecture also holds for N-Koszul algebras, graded local algebras, graded cellular algebras [4]. Our results show that the tame Hecke algebras of type **A** also provide a positive answer to Han's conjecture.

[References]

[1] AVRZMOV L L, VIGUÉ-POIRRIER M. Hochsschild homology criteria for smoothness[J]. Inter Math Res Not, 1992, 1: 17-25.

[2] BUCHWEITZ R O, GREEN E L, MADSEN D, et al. Finite Hochschild cohomology without finite global dimension[J]. Math Res Lett, 2005, 359: 805-816,

[3] BEILINSON A, GINSBURG V, SOERGEL W. Koszul duality patterns in representation theory[J]. J Amer Math Soc, 1996, 9: 473-527.

[4] BERGH P A, MADSEN D. Hochsschild homology and global dimension[J]. Bull London Math Soc, 2009, 41: 473-482.

[5] CARTAN H, EILENBERG S. Homological Algebra[M]. Princeton: Princeton University Press, 1956.

[6] ERDMANN K, NAKNO D K. Representation type of Hecke algebras of type A[J]. Trans Amer Math Soc, 2002, 354(1): 275-285.

[7] ERDMANN K, Schroll S. On the Hochschild cohomology of tame Hecke algebras[J]. Arch Math, 2010, 94: 117-127.

[8] GREEN E L, HARTMAN G, MARCOS E N, et al. Resolution over Koszul algebras[J]. Arch Math, 2005, 85: 118-127.

[9] HAN Y. Hochschild (co)homology dimension[J]. J London Math Soc, 2006, 73(2): 657-668.

[10] HAPPEL D. Hochschild Cohomology of Finite-dimensional Algebras[C]. New York: Springer, 1989: 108-126.

[11] IGUSA K. Notes on the no loop conjecture[J]. J Pure Appl Algebra, 1990, 69: 161-176.

[12] JOST T. Morita equivalence for blocks of Hecke algebras of symmetric groups[J]. J Algebra, 1997, 194: 201-223.

[13] LODAY J L. Cyclic Homology[M]. Berlin: Springer, 1992.

[14] MEMBRILO-HERNÁNEZ F H. Brauer tree algebras and derived equivalence[J]. J Pure Appl Algebra, 1997, 114: 231-258.

[15] OPDAM E, SOLLEVELD M. Homological algebra for affine Hecke algebras[J]. Adv Math, 2009, 220: 1549-1601.

[16] SCHROOL S, SNASHALL N. Hochschild cohomology and support varieties for tame Hecke algebras[J]. Quart J Math, 2011, 62: 1017-1029.

[17] SHEPLER A V, WITHERSPOON S. Hochschild cohomology and graded Hecke algebras[J]. Trans Amer Math Soc, 2008, 360(8): 3975-4005.

[18] XU Yun-ge, ZHAO Ti-wei. Hochschild cohomology rings of tame Hecke algebras[J]. Manus Math, 2013, 142(3-4): 491-512.