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Relatively Injective Modules with Respect to Torsion Theory

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Abstract: For a hereditary torsion theory τ, this paper mainly discuss properties of A-τ-injective modules, where A is a fixed left R-module. It is proved that if M is an A-τ-injective, B is a submodule of A, then 1) M is A/B-τ- injective; 2) M is B-τ-injective when B is τ-dense in A. Furthermore, we show that if $A_1, A_2, \cdots, A_n$ are relatively injective modules, then $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is self-τ-injective if and only if A_i is self-τ-injective for each i.

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§1. Introduction

Throughout this paper, R denotes an associative ring with non-zero identity and all modules are left unital R-modules. We denote by τ a hereditary torsion theory on the category R-Mod of left R-modules. τ-injective modules have been studied by many authors(e.g. see [1]~[4]). In the present paper, we are interested in study the properties of A-τ-injective modules.

Now, let us recall some basic notations and definitions.

A pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of left R-modules is called an hereditary torsion theory if it satisfy the following conditions 1) $\mathrm{Hom}_R(T, F) = 0$ for any $T \in \mathcal{T}$, $F \in \mathcal{F}$; 2) $\mathcal{T}$ is closed under submodules, homomorphic images, extensions and direct sums; 3) $\mathcal{F}$ is closed under

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injective hull, submodules, extensions and direct products. $\mathcal{T}$ is called τ-torsion class, $\mathcal{F}$ is called τ-torsionfree class(see [5]).

A module is said to be τ-injective if it is injective with respect to every monomorphism with it's cokernel in $\mathcal{T}$(see [1], [6]).

A submodule B of a module A is called τ-dense in A if A/B is in $\mathcal{T}$(see [6]).

A module is said to be A-injective if it is injective with respective to every monomorphism $f : B \longrightarrow A$(see [7]).

A module M is said to be self-injective if it is M-injective(see [7]).

Let us consider the notion of relative τ-injectivity.

Definition 1.1 Let A be a module. A module M is called A-τ-injective if for every τ-dense submodule B of A, $\mathrm{Hom}_R(A, M) \longrightarrow \mathrm{Hom}_R(B, M)$ is surjective.

A module M is called self-τ-injective if it is M-τ-injective.

Two modules A_1 and A_2 are said to be relatively τ-injective if A_1 is A_2-τ-injective and A_2 is A_1-τ-injective.

In this paper, suppose A is a fixed R-module, we prove that if M is an A-τ-injective, B is a submodule of A, then (1) M is A/B-τ-injective; (2) M is B-τ-injective when B is τ- dense in A. Also, if $A_1, A_2, \cdots, A_n$ are relatively injective modules, it is proved that $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is self-τ-injective if and only if A_i is self-τ-injective for each i.

§2. Main Results

Lemma 2.1 Let A be a module, and $(M_i)_{i\in I}$ be a family of modules. Then $\Pi_{i\in I} M_i$ is A-τ-injective if and only if M_i is A-τ-injective for every $i \in I$.

Proof Similar to the proof for A-injective modules(see [7]).

Theorem 2.2 Let A_1, A_2 be modules, $A = A_1 \oplus A_2$, and let p_1, p_2 be the canonical projections on A_1 and A_2 respectively. Then the following statements are equivalent:

(1) A_1 is A_2-τ-injective.

(2) For every submodule B of A such that $B \cap A_1 = 0$ and $p_2(B)$ is a τ-dense submodule of A_2, there exists a submodule C of A such that $A = A_1 \oplus C$ and $B \subseteq C$.

Proof (1) $\Longrightarrow$ (2) Suppose B is a submodule of A such that $B \cap A_1 = 0$ and $p_2(B)$ is a $\tau-$dense submodule of A_2. We define $f : B \longrightarrow p_2(B)$ given by $f(b) = p_2(b)$ for any $b \in B$. It is easy to see that f is an isomorphism since $B \cap A_1 = 0$. By hypothesis, the homomorphism $p_1 f^{-1} : p_2(B) \longrightarrow A_1$ extends to a homomorphism $g : A_2 \longrightarrow A_1$. Let

$$C = \{g(a) + a | a \in A_2\}.$$

Then C is a submodule of A. For any $a \in A$, $a = a_1 + a_2 = (a_1 - g(a_2)) + (a_2 + g(a_2))$, hence $A = A_1 + C$. For any $x \in A_1 \cap C$, $x = a_2 + g(a_2)$ for some $a_2 \in A_2$ and $x \in A_1$, then $a_2 = x - g(a_2) \in A_1 \cap A_2$, and so $a_2 = 0$, $x = 0$, therefore $A = A_1 \oplus C$. Since $b = p_1(b) + p_2(b) = p_1 f^{-1} p_2(b) + p_2(b) = gp_2(b) + p_2(b) \in C$ for any $b \in B$, we have $B \subseteq C$.

(2) $\Longrightarrow$ (1) Suppose M is a τ-dense submodule of A_2, $f: M \longrightarrow A_1$ is any homomorphism. Let

$$B = \{m - f(m) | m \in M\},$$

then B is a submodule of A and $B \cap A_1 = 0$. Also $p_2(B) = M$ is a τ-dense submodule of A_2. By hypothesis, there exists a submodule C of A such that $A = A_1 \oplus C$ and $B \subseteq C$. Let $p: A \longrightarrow A_1$ denote the projection of A with kernel C, and let $q: A_2 \longrightarrow A_1$ denote restriction of p to A_2. Then for any $m \in M$, we have

$$q(m) = p(m) = p(m - f(m)) + pf(m) = pf(m) = f(m).$$

Hence q extends f, and consequently A_1 is A_2-τ-injective.

Theorem 2.3 Let A be a module, M be an A-τ-injective module, B be a submodule of A. Then

(1) M is A/B-τ-injective.

(2) If B is τ-dense in A, then M is B-τ-injective.

Proof (1) Suppose T is a τ-dense submodule of A/B, there exists a submodule C of A such that $C/B = T$ and C is τ-dense in A. Let $f: C/B \longrightarrow M$ be any homomorphism. Denote by $i: C \longrightarrow A$ and $j: C/B \longrightarrow A/B$ the inclusions and by $p: C \longrightarrow C/B$ and $q: A \longrightarrow A/B$ the natural homomorphism. Since C is τ-dense in A and M is A-τ-injective, there exists a homomorphism $g: A \longrightarrow M$ such that $gi = fp$. It followings from $B = \ker p \subseteq \ker fp = \ker gi \subseteq \ker g$ that there is a homomorphism $h: A/B \longrightarrow M$ such that $g = hq$. For any $c \in C$,

$$hj(c + B) = hjp(c) = hqi(c) = fp(c) = f(c + B).$$

Thus M is A/B-τ-injective.

(2) Assume B is a τ-dense submodule of A. Let C be a τ-dense submodule of B and $f: C \longrightarrow M$ be a homomorphism. Then C is a τ-dense submodule of A. Denote by $i: C \longrightarrow B$ and $j: B \longrightarrow A$ inclusion homomorphisms. Since M is A-τ-injective, it has a homomorphism $g: A \longrightarrow M$ such that $gji = f$. Therefore the homomorphism $gj: B \longrightarrow M$ extends f. Thus M is B-τ-injective.

Corollary 2.4 Let A_1, A_2 be modules such that $A_1 \oplus A_2$ is self-τ-injective. Then A_1, A_2 are both self-τ-injective and relatively τ-injective.

Proof By Theorem 2.3 and Lemma 2.1.

Theorem 2.5 Let A_1 and A_2 be modules. If a module is A_1-τ-injective and A_2-injective, then it is $(A_1 \oplus A_2)$-τ-injective.

Proof Denote $A = A_1 \oplus A_2$. Let M be an A_1-τ-injective module and an A_2-injective module. Suppose B is a τ-dense submodule of A and $f: B \longrightarrow M$ is a homomorphism. Let μ denote the restriction of f to $B \cap A_1$, $i: B \longrightarrow A$ and $j: B \cap A_1 \longrightarrow A_1$the inclusion and $i_1 : A_1 \longrightarrow A$ denote the canonical injective. Since $A_1/(B \cap A_1) \cong (B + A_1)/B$ and $(B + A_1)/B \subseteq A/B$, $B \cap A_1$ is τ-dense in A_1. By hypothesis, it has a homomorphism $\nu : A_1 \longrightarrow M$ such that $\nu j = \mu$. Since A_1 is a direct summand of A. Denote $\omega = \nu p_1 : A \longrightarrow M$, then $\omega i_1 j = \nu p_1 i_1 j = \nu j = \mu$. Denote by g the restriction of ω to B. Since $B \cap A_1 \subseteq \ker(f - g)$, we can define a homomorphism $h : B/B \cap A_1 \longrightarrow M$ by $h(b + B \cap A_1) = f(b) - g(b)$ for any $b \in B$. Since $B \cap A_1 \subseteq A_1 = \ker p_2$, we can also define a homomorphism $\alpha : B/B \cap A_1 \longrightarrow A_2$ by $\alpha(b + B \cap A_1) = p_2(b)$. It is easy to see that α is a monomorphism. By hypothesis, M is A_2-injective, there exists a homomorphism $\beta : A_2 \longrightarrow M$ such that $\beta\alpha = h$. Let $\gamma = \omega + \beta p_2 : A \longrightarrow M$. For any $b \in B$,

$$\gamma i(b) = \omega(b) + \beta p_2(b) = \omega(b) + \beta\alpha(b + B \cap A_1) = \omega(b) + h(b + B \cap A_1) = f(b).$$

Therefore M is A-τ-injective.

Corollary 2.6 Let $A_1, A_2, \cdots, A_n$ be relatively injective modules. Then $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is self-τ-injective if and only if A_i is self-τ-injective for each i.

Proof It follows from Theorem 2.5 and Corollary 2.4.

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