Chin. Quart. J. of Math.
2009, 24 (1): 81—86

(Inco) Projective Modules of Relatively Hereditary Torsion Theroy

ZHANG Li-hong, XIN Da-wei

(*College of Mathematics, Jilin Normal University, Siping* 136000*, China*)

Abstract: In the paper, we define(inco) project modules of relatively hereditary torsion theory τ by intersection complement of module and study their properties; secondly, we define the(inco) τ-semisimple ring by(inco) τ-projective module and study their properties. When τ is a trivial torsion theory on R-mod, we prove that R is a semisimple ring if and only if R is a(inco) semisimple ring and satisfies(inco) condition.

Key words: Torsion theory τ; (inco) condition; τ-torsionfree module; (inco) τ-projective module; semisimple ring

2000 MR Subject Classification: 16S90

CLC number: O153.3 **Document code**: A

Article ID: 1002–0462 (2009) 01–0081–06

§1. Preliminary Knowledge

In the paper, R always denotes a ring with unit. All modules are left R-module and are unitary. The category of all left R-mod are denoted by R-mod. τ denotes a torsion theory on R-module. Let M be a left R-module and A be a submodule of M. In [1], we define an intersection complement of A in M. In the paper, by intersection complement we define the module satisfying(inco) condition and study relatively projective module satisfying(inco) condition, which is called(inco) τ-projective module. We also define(inco) τ-semisimple ring by(inco) τ-projective module and give some properties. When τ is a trivial torsion theory, R is a semisimple ring if and only if R is a (inco)τ-semisimple satisfying(inco) condition, thus we study more semisimple ring.

What we call torsion theories here will always be hereditary torsion theories, other notions involving torsion theory see [2].

Received date: 2004-12-20

Foundation item: Supported by the Science and Technology Develop Foundation of Jilin Science and Technology Department(20040506-3)

Biography: ZHANG Li-hong(1956-), male, native of Siping, Jilin, a professor of Jilin Normal University, engages in ring theory.

Definition 1 Let A be a submodule of left R-module M. A submodule A' of M is called an intersection complement, briefly inco, of A in M, if $A\cap A'$=0 and A' is maximal in $A\cap A'$=0 [1].

Definition 2 Let M be a left R-module. M is called satisfying(inco) condition, if M satisfies equivalent conditions of Exercise 11 in [1].

Lemma 1 If M satisfies(inco) condition, then so is any submodule of M.

Lemma 2 Let M be an injective module, then M satisfies(inco) condition if and only if M satisfies the maximal condition for direct summand.

Proof Since $M = E(M)$, according to Definition 2 and Exercise 11 of [1] M satisfies the maximal condition for direct summand. Conversely, if M satisfies the maximal condition for direct summand, then $E(M)$ satisfies the maximal condition for direct summand, thus M satisfies(inco) condition by Definition 2 and Exercise 11 of [1].

Lemma 3 Left R-module M satisfies(inco) condition if and only if $E(M)$ satisfies(inco) condition.

Proof If M satisfies(inco) condition, then $E(M)$ satisfies the maximal condition for direct summand by Definition 2 and Exercise 11 of [1]. According to Lemma 2, $E(M)$ satisfies(inco) condition. Conversely, if $E(M)$ satisfies(inco) condition, then $E(M)$ satisfies the maximal condition for direct summand by Lemma 2, thus M satisfies(inco) condition by Definition 2 and Exercise 11 of [1].

Definition 3 Let M be a left R-module. M is called(inco) τ-projective module, if for every left R-module exact sequence $B \xrightarrow{\pi} C \longrightarrow 0$ where B is τ-torsionfree module satisfying(inco) condition, every α: $M \longrightarrow C$, there exists $\beta : M \longrightarrow B$ with $\alpha = \beta\pi$.

It is clear that M is(inco) τ-projective module if and only if for short exact sequence $0\longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where B is τ-torsionfree module satisfying(inco) condition, there exists exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C) \longrightarrow 0.$$

§2. Main Results

Theorem 1 If τ is a torsion theory on R-mod, then the following conditions on a left R-module M are equivalent :

(i) $\mathrm{Ext}^1_R(M, N) = 0$ for every τ-torsionfree left R-module N satisfying(inco) condition;

(ii) If E is injective τ-torsionfree left R-module and satisfies(inco) condition, E_1 is a submodule of E, then there exists exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, E_1) \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(M, E/E_1) \longrightarrow 0;$$

(iii) M is(inco) τ-projective module.

Proof (i)$\Longrightarrow$(ii) Let E is injective τ-torsionfree left R-module and satisfies(inco) condition. If $E_1 \subseteq E$, then E_1 satisfies(inco) condition by Lemma 1. By Proposition 1.10 in [2], E_1 is a τ-torsionfree, therefore $\mathrm{Ext}_R^1(M, E_1) = 0$. By Theorem 7.3 in [3], proving (ii);

(ii)$\Longrightarrow$(iii) Consider the diagram

$$\begin{tikzcd}
 & & & M \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] \arrow[ddl, "\gamma"] & & \\
0 \arrow[r] & A \arrow[r] & B \arrow[rr, "\pi"] \arrow[d, "\lambda"'] & & C \arrow[r] \arrow[d, "\rho"] & 0 \\
 & & E(B) \arrow[rr, "\pi'"'] & & E(B)/A \arrow[r] & 0
\end{tikzcd}$$

in which B is τ-torsionfree left R-module and satisfies(inco) condition, λ is the canonical inclusion. We want a map $\beta : M \longrightarrow B$ with $\alpha = \beta\pi$. If $c \in C$, then there exists $b \in B \subseteq E(B)$ such that

$$c = \pi(b), \pi'(b) = b + A \in E(B)/A.$$

Let

$$\rho : C \longrightarrow E(B)/A, \ c \longrightarrow b + A$$

If c=0, then $b \in \ker \pi$=A, it follows that ρ is well defined R-morphism and $\rho\pi = \pi^{'}\lambda$.. If $\rho(c) = b + A = A$, then $b \in A$, $\pi(b) = c$=0, thus ρ is monomorphism. By Lemma 3 and Proposition 1.10 in [2], $E(B)$ satisfies(inco) condition and is τ-torsionfree module. Therefore, there exists $\gamma : M \longrightarrow E(B)$ with $\pi^{'}\gamma = \rho\alpha$.

If $x \in \mathrm{Im}\gamma$, then there exists $m \in M$ such that $x = \gamma(m)$, $\alpha(m) \in C$. Thus there exists $b \in B$ such that $\alpha(m) = \pi(b)$, we have

$$\rho\alpha(m) = \rho\pi(b) = \pi^{'}\lambda(b) = \pi^{'}(b) = \pi^{'}\gamma(m) = \pi^{'}(x),$$

it follows that $b - x \in \ker \pi^{'} = A \subseteq B$. Since $b \in B$, thus $x \in B$, i.e., $\mathrm{Im}\gamma \subseteq B$. By Theorem 3.5 in [4], there exists β: $M \longrightarrow B$ such that $\gamma = \lambda\beta$, thus

$$\pi^{'}\gamma = \pi^{'}\lambda\beta = \rho\pi\beta = \rho\alpha,$$

but ρ is injective, it follows that $\alpha = \pi\beta$;

(iii)$\Longrightarrow$(i) By Lemma 3 and Proposition 1.10 in [2], if N is a τ-torsionfree module satisfying(inco) condition, then so is $E(N)$. According to Theorem 7.3 and 7.6 in [3], we have a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, E(N)) \xrightarrow{\pi^*} \mathrm{Hom}_R(M, E(N)/N) \longrightarrow$$

$$\mathrm{Ext}_R^1(M, N) \longrightarrow Ext_R^1(M, E(N)) = 0$$

but $\pi^*$is a epimorphism, thus $\mathrm{Ext}_R^1(M, N) = 0$.

Theorem 2 Let τ be a torsion theory on the R-mod. Then following condition are equivalent :

(i) Every left R-module is(inco) τ-projective module;

(ii) Every simple module is(inco) τ-projective module;

(iii) If M is a τ-torsionfree module satisfying(inco) condition, then $J(M) = 0$;

(iv) Every τ-torsionfree module satisfying(inco) condition is semisimple;

(v) The class of all(inco) τ-projective module is closed under taking homomorphic images;

(vi) If N is a τ-torsionfree module satisfying(inco) condition and P is a projective module, P_1 is a submodule of P, then there exists exact sequence :

$$0 \longrightarrow \mathrm{Hom}_R(P/P_1, N) \longrightarrow \mathrm{Hom}_R(P, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow 0.$$

Proof (i)$\Longrightarrow$(ii) This is immediate;

(ii)$\Longrightarrow$ (iii) Let M be a τ-torsionfree module satisfying(inco) condition and let m be a nonzero element of M. Among those submodules of M not containing m there exists, by Zorn's Lemma, a maximal submodule N. Let N' be the intersection of all those submodules of M properly containing N.

If N is not a maximal submodule of M, then $N \subset N^{'} \neq M$, therefore $m \in N^{'}$ and N is a maximal submodule of N'. By Lemma 1 and Proposition 1.10 in [2], N' satisfies(inco) condition and is a τ-torsionfree module. By the hypothesis, $N^{'} \cong N \oplus N^{'}/N$, thus $N^{'}/N$ is a τ-torsionfree module.

If N' is a maximal submodule of M, then M/N' is a simple. By hypothesis, $M \cong N^{'} \oplus M/N^{'}$, thus $M/N^{'}$ is a τ-torsionfree module. By Proposition 1.12 in [2], M/N is a τ-torsionfree module. It is clear that N is a direct summand of $M, M \cong N \oplus N_1$, then $M/N \cong N_1$ satisfies(inco) condition. Therefore the short exact sequence

$$0 \longrightarrow N^{'}/N \longrightarrow M/N \longrightarrow M/N^{'} \longrightarrow 0$$

splits, this is, there exists a submodule $M^{'}$ of M such that $M/N \cong N^{'}/N \oplus M^{'}/N$. But $m \in N^{'}$, $m \notin N$, so $m \notin M^{'}$. It follows that $M^{'} = N, M = N^{'}$ contradicting the maximality of $N^{'}$. This proves that $N^{'}$ is not a maximal submodule of M.

Assume $N^{''} = \cap\{K | K$ is submodule of M, $K \supset N'\}$, it is clear that $N^{''}/N'$ is a simple module and $N^{''}$ is a τ-torsionfree module satisfying(inco) condition. By given, $N^{''} \cong N' \oplus N^{''}/N'$. Consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N'' & \longrightarrow & M & \xrightarrow{\pi_1} & M/N'' & \longrightarrow & 0 \\ & & \downarrow & & \pi_2\downarrow & & 1\downarrow & & \\ 0 & \longrightarrow & N''/N & \longrightarrow & M/N & \xrightarrow{\pi_3} & M/N'' & \longrightarrow & 0. \end{array}$$

If $N^{''}$ is a maximal submodule of M, then M/$N^{''}$ is a simple, thus by the hypothesis split of the top row gives split of the bottom. Hence there exists a submodule $M^{''}$ of M such that M/N$\cong$ $N^{''}/N' \oplus M^{''}/N$. Since m$\in$ $N^{''}$, $m \notin N$, so m$\notin$ $M^{''}$. By the hypothesis on N, this implies $M^{''} = N$, $M = N^{''}$ contradicting the maximality of $N^{''}$. This proves that $N^{''}$ is not a maximal submodule of M. This argument yields an ascending chain

$$N \subset N' \subset N^{''} \subset \cdots$$

of submodules of M and the former is a direct summand of the latter, contradicting the assumption of M. It follows that N is a maximal submodule of M which does not contain m. This implies that m does not belong to $J(M)$. Since m is an arbitrary nonzero element of M, we conclude that $J(M) = 0$;

(iii)$\Longrightarrow$(iv) Let M be a τ-torsionfree left R-module satisfying(inco) condition and let $N1$ be a simple submodule of M. Since $J(M) = 0$, there exists a maximal submodule of M such that $N1 \not\subseteq M_1$, thus $M = N_1 \oplus M_1$. We know that M_1 is also a τ-torsionfree module and satisfies(inco) condition. By Lemma 5.1.3 in [1], every small submodule in M_1 is also small in M, hence J(M_1)$\subseteq$J(M)=0. By aforesaid process, M_1= $N_2 \oplus M_2$, in which N_2 is a simple module and M_2 is a τ-torsionfree module satisfying(inco) condition and J(M_2)$\subseteq$J(M_1)=0. This argument yields an expression

$$M \cong N_1 \oplus N_2 \oplus \cdots \oplus N_k \oplus \cdots$$

Let C_1=N_1, C_k=$N_1 \oplus N_2 \oplus \ldots \oplus N_k$. Then there is an ascending chain

$$C_1 \subset C_2 \subset \cdots \subset C_k \subset \cdots,$$

where C_i is a direct summand of C_{i+1}. Since M satisfying(inco) condition, proving (iv);

(iv)$\Longrightarrow$(i) Let M be a left R-module. Consider the short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow 0$$

with B τ-torsionfree and satisfying(inco) condition. By hypothesis, B is a semisimple module, hence the exact sequence splits. If α: $M \longrightarrow C$ is any map, then there exists $\beta: M \longrightarrow B$ with $\alpha = \pi\beta$, proving (i);

(i)$\Longrightarrow$(v) This is immediate;

(v)$\Longrightarrow$(vi) Let P be a projective module and let P_1 be a submodule of P. If N is a τ-torsionfree module satisfying(inco) condition, then there exists a long exact chain

$$0 \longrightarrow \mathrm{Hom}_R(P/P_1, N) \longrightarrow \mathrm{Hom}_R(P, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow$$

$$\mathrm{Ext}^1_R(P/P_1, N) \longrightarrow \cdots.$$

Since P/P_1 is a(inco)τ-projective module, by Proposition 1. $\mathrm{Ext}^{'}_R(P/P_1, N) = 0$ proving (vi);

(vi)$\Longrightarrow$(i) Let M be a left R-module. Then there is a projective module P and an exact sequence

$$0 \longrightarrow P_1 \longrightarrow P \longrightarrow M \longrightarrow 0.$$

If N is a τ-torsionfree module and satisfies(inco) condition, then there exists a long exact chain

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(P,N) \longrightarrow \mathrm{Hom}_R(P_1,N) \longrightarrow$$

$$\mathrm{Ext}_R^1(M,N) \longrightarrow \mathrm{Ext}_R^1(P,N) = 0.$$

By hypothesis, $\mathrm{Ext}_R^1(M,N) = 0$, this implies that M is a(inco) module by Proposition 1.

Definition 4 R is called(inco) τ-semisimple ring, if R satisfies equivalent condition of Proposition 2.

When τ is a trivial torsion theory on R-module, every left R-module is all τ-torsionfree module.

Theorem 3 Let τ be a trivial torsion theory. Then R is a semisimple ring if and only if R satisfies(inco) condition and is(inco) τ-semisimple ring.

Proof This is immediate. Conversely, by Proposition 2, if R is a(inco) τ-semisimple ring and τ is a trivial torsion theory, every left R-module satisfying(inco) condition is semisimple module, hence as a left R-module, R is semisimple, i.e., R is a semisimple ring.

[References]

[1] KASCH F. Modules and Rings[M]. New York: Academic Press, Inco(London) LTD, 1982.

[2] GOLAN J S. Torsion Theories[M]. New York: Copublished in the United States with John Wileg Sons Inco, 1986.

[3] ROTMAN J. An Introduction to Homological Algebra[M]. New York: Academic Press, 1979.

[4] BLYTH T S. Module Theory[M]. London: Oxford University Press, 1977.