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On Variational Inequality and Fixed Point Problems

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Abstract: In this paper, A strong convergence theorem for a finite family of nonexpansive mappings and relaxed cocoercive mappings based on an iterative method in the framework of Hilbert spaces is established.

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§1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let C be a nonempty closed and convex subset of H and let $A: C \to H$ be a nonlinear mapping. The classical variational inequality problem, which denoted by $VI(C, A)$, is to find $u \in C$ such that $\langle Au, v-u\rangle \geq 0$ for all $v \in C$. A is said to be relaxed (u, v)-cocoercive if there exist two constants $u,\ v > 0$ such that $\langle Ax - Ay, x - y\rangle \geq (-u)\|Ax - Ay\|^2 + v\|x - y\|^2$, for all $x,\ y \in C$. Recall that a mapping $T: C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in C$. The fixed point of T is denoted by $F(T)$.

Iterative methods are efficient tool in nonlinear analysis and optimization theory. Many well-known problems arising in various branches of science can be studied by using algorithms which are iterative in their nature; see, for example, [1-4].

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In this paper, an iterative algorithm is considered for the the problem of finding a common element in the fixed point set of nonexpansive mappings and in the solution set of variational inequalities. Define a mapping W_n by: $U_{n0} = I$ and

$$\begin{aligned} &U_{n1} = \lambda_{n1} T_1 U_{n0} + (1-\lambda_{n1}) I, \\ &U_{n2} = \lambda_{n2} T_2 U_{n1} + (1-\lambda_{n2}) I, \\ &\cdots, \\ &W_n = U_{nN} = \lambda_{nN} T_N U_{n,N-1} + (1-\lambda_{nN}) I, \end{aligned} \tag{1.1}$$

where $\{\lambda_{n1}\}, \{\lambda_{n2}\}, \cdots, \{\lambda_{nN}\} \in (0,1]$. Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \cdots, T_N$ and $\{\lambda_{n1}\}, \{\lambda_{n2}\}, \cdots, \{\lambda_{nN}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. It is proved that $F(W_n) = \bigcap_{i=1}^{N} F(T_i)$; see [1] for more details.

Based on the mapping W_n, a strong convergence theorem of common elements is established. In order to prove our main results, we need the following lemmas.

Lemma 1.1[5] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf\limits_{n\to\infty} \beta_n \le \limsup\limits_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n) y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup\limits_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.2[6] Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$. Then $\lim_{n\to\infty} \alpha_n = 0$.

§2. Main Results

Theorem 2.1 Let $A : C \to H$ be be a relaxed (u,v)-cocoercive and μ-Lipschitz continuous mapping. Let $f : C \to C$ be a contraction with the coefficient α, where $0 < \alpha < 1$ and T_i, where $i = 1, 2, \cdots, N$, a nonexpansive mapping. Assume that $\Upsilon = \bigcap_{i=1}^{N} F(T_i) \bigcap VI(C,A) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ a sequence generated in the following algorithm

$$x_{n+1} = \alpha_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n P_C (I - r_n A) P_C (I - s_n A) x_n, \quad n \ge 1,$$

where W_n is defined by (1.1), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{r_n\}$ and $\{s_n\}$ are chosen such that $\alpha_n + \beta_n + \gamma_n = 1$; $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \liminf\limits_{n\to\infty} \beta_n \le \limsup\limits_{n\to\infty} \beta_n < 1$; $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$; $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$; $\{r_n\}$, $\{s_n\} \in [a,b]$ for some a, b with $0 < a < b < \frac{2(v-u^2)}{2}$; $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$, for all $i = 1, 2, \cdots, N$. Then $\{x_n\}$ converges strongly to $q \in \Upsilon$, where $q = P_\Upsilon f(q)$.

Proof Since A is relaxed (u,v)-cocoercive and Lipschitz continuous, we see that $I - r_n A$ and $I - s_n A$ are nonexpansive. Letting $p \in \Upsilon$, we arrive at

$$\|x_{n+1} - p\| \le [1 - \alpha_n (1-\alpha)] \|x_n - p\| + \alpha_n \|p - f(p)\|,$$

which implies that the sequence $\{x_n\}$ is bounded. Notice that

$$\|x_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + M_1 (|s_{n+1} - s_n| + |r_{n+1} - r_n|),$$

where M_1 is an appropriate constant. Putting $l_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ and $y_n = P_C(I - r_nA)P_C(I - s_nA)x_n$, we see that

$$\begin{aligned}&\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\ \le\ & M_2(|s_{n+1} - s_n| + |r_{n+1} - r_n| + 2\sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|) \\ &+ (\|f(W_nx_n)\| + \|W_ny_n\|)|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|,\end{aligned}$$

where M_2 is an appropriate constant. It follows that $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0$. In view of Lemma 1.1, we see that $\lim_{n\to\infty} \|l_n - x_n\| = 0$. It follows that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. On the other hand, we have

$$x_{n+1} - x_n = \alpha_n(f(W_nx_n) - x_n) + \gamma_n(W_ny_n - x_n).$$

It follows that $\lim_{n\to\infty} \|W_ny_n - x_n\| = 0$. Put $z_n = P_C(I - s_nA)x_n$. For each $p \in \Upsilon$, we see that

$$\|z_n - p\|^2 \le \|x_n - p\|^2 + (2s_nu + s_n^2 - \frac{2s_nv}{\mu^2})\|Ax_n - Ap\|^2$$

and

$$\|y_n - p\|^2 \le \|x_n - p\|^2 + (2r_nu + r_n^2 - \frac{2r_nv}{v^2})\|Az_n - Ap\|^2.$$

On the other hand, we have

$$\|x_{n+1} - p\|^2 \le \alpha_n\|f(W_nx_n) - p\|^2 + \|x_n - p\|^2 + \gamma_n(2r_nu + r_n^2 - \frac{2r_nv}{\mu^2})\|Az_n - Ap\|^2.$$

This implies that $\lim_{n\to\infty} \|Az_n - Ap\| = 0$. In a similar way, we can show that $\lim_{n\to\infty} \|Ax_n - Ap\| = 0$. In view of $\lim_{n\to\infty} \|W_ny_n - x_n\| = 0$,

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|z_n - y_n\|^2 + 2r_n\|z_n - y_n\|\|Az_n - Ap\|$$

and

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2s_n\|x_n - z_n\|\|Ax_n - Ap\|,$$

we can obtain that $\lim_{n\to\infty} \|W_nx_n - x_n\| = 0$. Observe that P_Cf is a contraction. Indeed, for $\forall x,\ y \in C$, we have

$$\|P_f(x) - P_Cf(y)\| \le \|f(x) - f(y)\| \le \alpha\|x - y\|.$$

Banach's contraction mapping principle guarantees that has a unique fixed point, say $q \in H$. That is $q = P_Ff(q)$.

Next, we show $\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle \le 0$. To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \lim_{n\to\infty}\langle f(q) - q, x_{n_i} - q\rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that there is a sequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume, without loss of generality, that $x_{n_i} \rightharpoonup p$. It follows that $z_{n_i} \rightharpoonup p$. Hence we have $p \in \Upsilon$. Indeed, let us first show that $p \in VI(C, A)$. Put

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1, & w_1 \in C, \\ \emptyset, & w_1 \notin C. \end{cases}$$

Since A is relaxed (u, v)-cocoercive and μ-Lipschitz continuous, we have

$$\langle Ax - Ay, x - y \rangle \geq (-u)\|Ax - Ay\|^2 + v|x - y\|^2 \geq (v - u\mu^2)\|x - y\|^2 \geq 0,$$

which yields that A is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_C w_1$ and $y_n \in C$, one has $\langle w_1 - y_n, w_2 - Aw_1 \rangle \geq 0$. On the other hand, we see from $z_n = P_C(I - s_n A)x_n$ that $\langle w_1 - z_n, z_n - (I - s_n A)x_n \rangle \geq 0$ and hence $\langle w_1 - z_n, \frac{z_n - x_n}{s_n} + Ax_n \rangle \geq 0$. It follows that

$$\begin{aligned}
&\langle w_1 - z_{n_i}, w_2 \rangle \\
\geq\ &\langle w_1 - z_{n_i}, Aw_1 \rangle \\
\geq\ &\langle w_1 - z_{n_i}, Aw_1 - \frac{z_{n_i} - x_{n_i}}{s_{n_i}} - Ax_{n_i} \rangle \\
=\ &\langle w_1 - z_{n_i}, Aw_1 - Az_{n_i} \rangle - \langle w_1 - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle + \langle w_1 - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{s_{n_i}} \rangle \\
\geq\ &\langle w_1 - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \langle w_1 - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{s_{n_i}} \rangle,
\end{aligned}$$

which implies that $\langle w_1 - p, w_2 \rangle \geq 0$. We have $p \in T^{-1}0$ and hence $p \in VI(C, A)$.

Next, let us show $p \in \bigcap_{i=1}^N F(T_i)$. Since Hilbert spaces are Opial's spaces, we have

$$\begin{aligned}
&\liminf_{i\to\infty} \|x_{n_i} - p\| \\
<\ &\liminf_{i\to\infty} \|x_{n_i} - W_n p\| \\
=\ &\liminf_{i\to\infty} \|_{n_i} - W_n x_{n_i} + W_n x_{n_i} - W_n p\| \\
\leq\ &\liminf_{i\to\infty} \|W_n x_{n_i} - W_n p\| \\
\leq\ &\liminf_{i\to\infty} \|x_{n_i} - p\|,
\end{aligned}$$

which derives a contradiction. Thus we have $p \in \bigcap_{i=1}^N F(T_i)$. This in turn implies that

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q \rangle = \lim_{n\to\infty} \langle f(q) - q, x_{n_i} - q \rangle = \langle f(q) - q, p - q \rangle \leq 0.$$

On the other hand, one has

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
\leq\ &\|\beta_n(x_n - q) + \gamma_n(W_n y_n - q)\|^2 + 2\alpha_n \langle f(W_n x_n) - q, x_{n+1} - q \rangle \\
\leq\ &(1 - \alpha_n)^2 \|x_n - p\|^2 + \alpha\alpha_n(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\
&+ 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle.
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\|x_{n+1}-q\|^2\\
\le&\frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}\|x_n-q\|^2-\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q,x_{n+1}-q\rangle\\
\le&\frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n-q\|^2-\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q,x_{n+1}-q\rangle+M_3\alpha_n^2\\
=&(1-l_n)\|x_n-q\|^2+\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}t_n,
\end{aligned}
$$

where M_3 is an appropriate constant,

$$l_n=\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \text{ and } t_n=\frac{M_3(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)}+\frac{1}{1-\alpha}\langle f(q)-q,x_{n+1}-q\rangle.$$

In view of Lemma 1.2, we can conclude the desired conclusion immediately. This completes the proof.

If $f(x)=u$, where $u\in C$ is a fixed element, then we have the following Halpern-type iterative algorithm.

Corollary 2.1 Let $A:C\to H$ be be a relaxed (u,v)-cocoercive and μ-Lipschitz continuous mapping. Let T_i, where $i=1,2,\cdots,N$, be a nonexpansive mapping. Assume that $\Upsilon=\bigcap_{i=1}^N F(T_i)\bigcap VI(C,A)\neq\emptyset$. Let $x_1\in C$ and $\{x_n\}$ be a sequence generated in the following algorithm

$$x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n W_n P_C(I-r_nA)P_C(I-s_nA)x_n,\quad n\ge 1,$$

where W_n is defined by (1.1), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{r_n\}$ and $\{s_n\}$ are chosen such that $\alpha_n+\beta_n+\gamma_n=1$; $\lim_{n\to\infty}\alpha_n=0$; $\sum_{n=1}^\infty\alpha_n=\infty$; $0<\liminf\limits_{n\to\infty}\beta_n\le\limsup\limits_{n\to\infty}\beta_n<1$; $\lim_{n\to\infty}|r_{n+1}-r_n|=0$; $\lim_{n\to\infty}|s_{n+1}-s_n|=0$; $\{r_n\},\{s_n\}\in[a,b]$ for some $a,\ b$ with $0<a<b<\frac{2(v-u^2)}{2}$; $\lim_{n\to\infty}|\lambda_{n,i}-\lambda_{n-1,i}|=0$, for all $i=1,2,\cdots,N$. Then $\{x_n\}$ converges strongly to $q\in\Upsilon$, where $q=P_\Upsilon u$.

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