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The Excellent Extensions of Rings and Copure Dimensions

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Abstract: Let R be a noetherian ring and S an excellent extension of R. $\mathrm{cid}(M)$ denotes the copure injective dimension of M and $cfd(M)$ denotes the copure flat dimension of M. We prove that if M_S is a right S-module then $\mathrm{cid}(M_S) = \mathrm{cid}(M_R)$ and if ${}_SM$ is a left S-module then $cfd({}_SM) = cfd({}_RM)$. Moreover, cid-$D(S)$ = cid-$D(R)$ and cfd-$D(S) = cfd$-$D(R)$.

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§1. Introduction

Throughout this article, all rings are associative with identity and all modules are unitary modules unless stated otherwise. $\mathrm{cid}(M)$(resp $cfd(M)$) denotes the copure injective dimension(resp copure flat dimension) of M. cid-$D(R)$(resp cfd-$D(R)$) denotes the copure injective dimension(resp copure flat dimension) of R. If A is a ring and N_A is a submodule of M_A, the notation $N_A|M_A$ means that there is an A-submodule P_A such that $M_A = N_A \oplus P_A$.

Excellent extension is an important extension of rings. It contains full matrix rings over ring R ([1]) and the ring $R * G$ of finite group G with $|G|^{-1} \in R$ ([2]). It was proved in [3] that if S is an excellent extension of R, then S and R have the same right global dimensions and weak global dimensions, the author considered the FP-injective dimensions in [4] and proved that if M_S is a right S-module, then FP-$id(M_S) = FP$-$id(M_R)$ and FP-$D(S) = FP$-$D(R)$. In this article, we consider the copure injective dimension and copure flat dimension.

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We proved that if R is a noetherian ring, S is an excellent extension of R and M_S is an S-module, then $\mathrm{cid}(M_S) = \mathrm{cid}(M_R)$; if ${}_SM$ is a S-module, then $cfd({}_SM) = cfd({}_RM)$. Moreover, we obtain cid-$D(S) =$ cid-$D(R)$ and cfd-$D(S) = cfd$-$D(R)$.

§2. Preliminaries

In order to consider the copure injective dimension and copure flat dimension, we first give some definitions and some results that are well known.

Definition 1.1 Let R be a ring. We call a right R-module M copure injective if for any injective right R-module E, $\mathrm{Ext}_R^1(E, M) = 0$.

Definition 1.2 Let R be a ring. We call a right R-module M strongly copure injective if for any injective right R-module E and $i \geq 1$, $\mathrm{Ext}_R^i(E, M) = 0$.

Definition 1.3 Let R be a ring. We call a left R-module M copure flat if for any injective right R-module E, $\mathrm{Tor}_1^R(E, M) = 0$.

Definition 1.4 Let R be a ring. We call a left R-module M strongly copure flat if for any injective right R-module E and $i \geq 1$, $\mathrm{Tor}_i^R(E, M) = 0$.

Let R be a ring with identity, S be an extension of R, where S and R have the same identity 1. The ring S is called an excellent extension of R if

(1) S is a free normalizing extension of R with a basis that includes 1; that is, there is a finite set $\{a_1, \cdots, a_n\} \subseteq S$ such that a_1=1, $S = Ra_1 + \cdots + Ra_n$, $a_iR = Ra_i$, for all $i = 1, \cdots, n$ and S is free with basis $\{a_1, \cdots, a_n\}$ as both a right and left R-module and

(2) S is R-projective; that is, if N_S is a submodule of M_S. then $N_R|M_R$ implies $N_S|M_S$.

Lemma 1.5[4] Let M be an R-module, then $cfdM = \mathrm{cid}M^+$.

Lemma 1.6[4] Let R be a noetherian ring. Then for an R-module M, the following statements are equivalent:

(1) $\mathrm{cid}M \leq n$;

(2) For any injective R-module E and $i \geq n+1$, $\mathrm{Ext}^i(E, M) = 0$;

(3) The nth cosyzygy of M is strongly copure injective.

Lemma 1.7[4] Let R be a ring. Then for an R-module M, the following statements are equivalent:

(1) $cfdM \leq n$;

(2) For any injective right R-module E and $i \geq n+1$, $\mathrm{Tor}_i(E, M) = 0$;

(3) The nth syzygy of M is strongly copure flat.

General background materials can be found in [1-10].

§3. Main Results

Theorem 2.1 Let R be a noetherian ring and S an excellent extension of R. If M is a right S-module, then $\mathrm{cid}(M_S) = \mathrm{cid}(M_R)$.

Proof If $\mathrm{cid}(M_S) = \infty$, then $\mathrm{cid}(M_R) \leq \mathrm{cid}(M_S)$.

Let $\mathrm{cid}(M_S) = m < \infty$. Since ${}_RS$ is a flat left R-module, then for any right R-module X, we have

$$\mathrm{Ext}_S^{m+j}(X \bigotimes_R S, M) \simeq \mathrm{Ext}_R^{m+j}(X, M),\ j \geq 1.$$

If X_R is injective, then $X \bigotimes_R S_S$ is injective by Proposition 7 of [5]. So $\mathrm{Ext}_S^{m+j}(X \bigotimes_R S, M)$ $= 0,\ j \geq 1$. And thus $\mathrm{Ext}_R^{m+j}(X, M) = 0$, that is, $\mathrm{cid}(M_R) \leq m$. Therefore, $\mathrm{cid}(M_R) \leq \mathrm{cid}(M_S)$.

Conversely, suppose that $\mathrm{cid}(M_R) = m < \infty$. For any right S-module X, since S_R is projective and ${}_SS$ is flat, we have natural isomorphism:

$$\mathrm{Ext}_R^{m+j}(X, M) \simeq \mathrm{Ext}_R^{m+j}(X \bigotimes_S S, M) \simeq \mathrm{Ext}_S^{m+j}(X, \mathrm{Hom}_R(S, M)).$$

If X_S is an injective right S-module, Then $\mathrm{Ext}_R^{m+j}(X, M) = 0$ since X_R is an injective right R-module by [6]. And so $\mathrm{Ext}_S^{m+j}(X, \mathrm{Hom}_R(S, M)) = 0$. There exists a right S-module Q_S such that $M_S \oplus Q_S \simeq \mathrm{Hom}(S, M)_S$ by [7]. Thus we have

$$\mathrm{Ext}_S^{m+j}(X, M) \oplus \mathrm{Ext}_S^{m+j}(X, Q) \simeq \mathrm{Ext}_S^{m+j}(X, \mathrm{Hom}_R(S, M)) = 0.$$

And so $\mathrm{Ext}_S^{m+j}(X, M) = 0$. Therefore, $\mathrm{cid}(M_S) \leq m$.

If $\mathrm{cid}(M_R) = \infty$, it is clear that $\mathrm{cid}(M_S) \leq \mathrm{cid}(M_R)$. Therefore, $\mathrm{cid}(M_S) = \mathrm{cid}(M_R)$.

Corollary 2.2 Let R be a noetherian ring and S an excellent extension of R. If M is a right S-module, then M_S is strongly copure injective if and only if M_R is strongly copure injective.

Corollary 2.3 Let R be a noetherian ring and S an excellent extension of R. Then the following statements are equivalent:

(1) Every left ideal of R is strongly copure injective;

(2) Every left ideal of S is strongly copure injective;

(3) Every right ideal of R is strongly copure injective;

(4) Every right ideal of S is strongly copure injective.

For a ring R. We define the right copure injective dimension of R. We denote cid-$D(R)$

$$\text{cid-}D(R) = \sup\{\ \mathrm{cid}(M_R) \mid M \text{ is a right } R\text{-module}\ \}.$$

It is clear that cid-$D(R) \le D(R)$, where $D(R)$ denotes the right global dimension of R.

Theorem 2.4 Let R be a noetherian ring and S an excellent extension of R. Then cid-$D(S) = cid$-$D(R)$.

Proof From Theorem 2.1, it follows that cid-$D(S) \le$ cid-$D(R)$.

If cid-$D(S) = \infty$, it is clear that cid-$D(R) \le$ cid-$D(S)$.

Let cid-$D(S) = m < \infty$. Then for any right R-module M and any right S-module X, we get

$$\mathrm{Ext}_R^{m+j}(X \bigotimes_S S, M) \simeq \mathrm{Ext}_S^{m+j}(X, \mathrm{Hom}_R(S, M)),\ j \ge 1.$$

Since $\mathrm{cid}(\mathrm{Hom}_R(S, M)_S) \le$ cid-$D(S)$=m, then for any injective right S-module X, we get

$$\mathrm{Ext}_S^{m+j}(X, \mathrm{Hom}_R(S, M)) = 0, j \ge 1.$$

And hence $\mathrm{Ext}_R^{m+j}(X, M) = 0,\ j \ge 1$, where X is an injective right S-module.

If Y is an injective right R-module. Then $Y \bigotimes_R S$ is an injective right S-module by Proposition 7 of [5]. So $\mathrm{Ext}_R^{m+j}(Y \bigotimes_R S, M) = 0$. Since S is an excellent extension of R, as right R-modules, we have

$$Y \bigotimes_R S_R = Y \bigotimes_R (\bigoplus_{i=1}^n Rs_i) \simeq Y \bigotimes_R (\bigoplus_{i=1}^n R) \simeq \bigoplus_{i=1}^n (Y \bigotimes_R R) \simeq \bigoplus_{i=1}^n Y.$$

Thus we have

$$\bigoplus_{i=1}^n \mathrm{Ext}_R^{m+j}(Y, M) \simeq \mathrm{Ext}_R^{m+j}(\bigoplus_{i=1}^n Y, M) \simeq \mathrm{Ext}_R^{m+j}(Y \bigotimes_R S, M) = 0,\ j \ge 1.$$

And hence $\mathrm{Ext}_R^{m+j}(Y, M) = 0,\ j \ge 1$. That is, $\mathrm{cid}(M_R) \le m$. Then cid-$D(R) \le m$ and so cid-$D(R) \le$ cid-$D(S)$. Therefore,

$$\text{cid-}D(R) = \text{cid-}D(S).$$

Theorem 2.5 Let R be a noetherian ring and S an excellent extension of R. If M is a left S-module, then $cfd(_SM) = cfd(_RM)$.

Proof From Lemma 1.5, it follows that $cfd(_RM) = \mathrm{cid}(M_R^+)$ and $cfd(_SM) = \mathrm{cid}(M_S^+)$. By Theorem 2.1, we have $\mathrm{cid}(M_R^+) = \mathrm{cid}(M_S^+)$. Hence $cfd(_RM) = cfd(_SM)$.

Corollary 2.6 Let R be a noetherian ring and S an excellent extension of R. If M is a left S-module, then $_SM$ is strongly copure flat if and only if $_RM$ is strongly copure flat.

Corollary 2.7 Let R be a noetherian ring and S an excellent extension of R. Then the following statements are equivalent:

(1) Every left ideal of R is strongly copure flat;

(2) Every left ideal of S is strongly copure flat;

(3) Every right ideal of R is strongly copure flat;

(4) Every right ideal of S is strongly copure flat.

For a ring R. We define the right copure flat dimension of R. We denote $cfd\text{-}D(R)$

$$cfd\text{-}D(R) = \sup\{\ \mathrm{cid}(M_R) \mid M \text{ is a left } R\text{-module }\}.$$

It is clear that $cfd\text{-}D(R) \leq WD(R)$, where $WD(R)$ denotes the left weak global dimension of R.

Theorem 2.8 Let R be a noetherian ring and S an excellent extension of R. Then cid-$D(S) =$ cid-$D(R)$.

Proof From Theorem 2.5, it follows that $cfd\text{-}D(S) \leq cfd\text{-}D(R)$.

If $cfd\text{-}D(S) = \infty$, then it is clear that $cfd\text{-}D(R) \leq cfd\text{-}D(S)$.

Let $cfd\text{-}D(S) = m < \infty$. Then for any left R-module M $cfd({}_RM) = \mathrm{cid}(M_R^+)$ by Lemma 1.5. By Theorem 2.1, we have $\mathrm{cid}(M_R^+) = \mathrm{cid}(M_S^+)$. And $\mathrm{cid}(M_S^+) = cfd({}_SM)$ by Lemma 1.5. So $cfd({}_RM) = cfd({}_SM) \leq m$, that is, $cfd\text{-}D(R) \leq cfd\text{-}D(S)$. Therefore, $cfd\text{-}D(S) = cfd\text{-}D(R)$.

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