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Virtual Element Method of the Allen-Cahn Equation

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Abstract: In this article, the virtual element method of the Allen-Cahn equation on a polygon grid is discussed in the fully discrete formulation. With the help of the energy projection operator, we give the corresponding error estimates in the L^2 norm and H^1 norm.

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§1. Introduction

Allen-Cahn equation is an important model proposed by ALLEN and CAHN in [3] for describing the antiphase boundary motion in crystals. In materials science, it is often used in the study of fluid dynamics and reaction-diffusion problems, but also widely used in the competition and repulsion of biological populations, river bed migration process.

We consider the following Allen-Cahn equation

$$\begin{cases} u_t - \varepsilon\Delta u + f(u) = 0, \quad (X,t)\in\Omega\times J, \\ u = 0, \quad (X,t)\in\partial\Omega\times J, \\ u(X,0) = u_0, \quad X\in\Omega, \end{cases} \tag{1.1}$$

with the energy of

$$E(u) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + F(u)\right)dX,$$

where u represents the concentration of one of the two metallic components of the alloy, $\varepsilon > 0$ represents the inter-facial width, $\Omega\in R^2$ denotes a bounded convex polygonal domain with

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boundary $\partial\Omega$. We shall use the notation n for a unit normal of an edge, $J=[0,T]$. $f(u)=F^{'}(u)$ where $F(u)=(u^2-1)^2/4$ represents the energy potential. Obviously, the continuous function $f(u)$ satisfies the Lipschitz condition.

The virtual element method (VEM) is a generalization of the classical finite element method to polygonal or polyhedral meshes. The main advantages of the virtual element method are the flexibility in mesh handing and the property of avoiding the explicit construction of the shape function. Conforming VEM was first introduced in [2,16,18]. Then many investigations have been conducted for VEM [1,4,6,14,15,17,20].

There are many excellent works on the convergent analysis for Allen-Cahn equation [5,7–13, 19]. The main of this paper is to develop the first-order implicit-explicit scheme by the virtual element method for Allen-Cahn equation. The error estimates in the L^2 norm and H^1 norm are derived.

The arrangement of this paper is as follows: Section 2 gives the conforming virtual element space. Section 3 constructs the variational form and gives the fully discrete form. In Section 4, the error estimates for the L^2 norm and H^1 norm corresponding to the fully discrete scheme are presented.

In this paper, we adopt standard notations for the Lebesgue spaces $L^p(\Omega)$, $1\le p\le\infty$ and the Sobolev Spaces $H^s(\Omega)$, $s\ge 0$ and their associated norms $\|\cdot\|_{s,\Omega}$ and seminorms $|\cdot|_{s,\Omega}$. The Banach space of all L^p integrable functions $\psi(t)$:$[0,T]\to H^s(\Omega)$ with norm $\|\psi\|_{L^p(H^s)}:=\left(\int_0^T\|\psi\|_{s,\Omega}^p\right)^{1/p}$ for $p\in[1,\infty)$ and standard modification at $p=\infty$. The generic positive constant will be denoted by C, which is independent of the mesh size h, but may depend on the size of Ω and take different values at different places.

§2. Conforming virtual element

Let T_h be a sequence of decompositions of Ω into general polygons. h denotes the maximum of the diameters of the elements in T_h, ε_h represents the set of edges, $n(K)$ denotes the number of vertexes of element K. Let $k\ge 1$ denote the "polynomial degree" of the methodology. For all $K\in T_h$, the augmented local space $V^k(K)$ is defined by

$$V^k(K)=\left\{v\in H^1(K)\cap C^0(\partial K): v|_e\in\mathbb{P}_k(e),\ \forall e\in\partial K,\ \Delta v\in\mathbb{P}_k(K)\right\}.$$

Similar to reference [15], we define the d.o.f. as

- Values of v at $n(K)$ vertexes of K.
- Values of v at $k-1$ uniformly spaced points on each edge e.
- All moments $\int_K vpdx$, for all $p\in\mathbb{P}_{k-2}(K)$.

We define the projection operator $\Pi_{k,K}^{\nabla}$: $H^1(K)\to\mathbb{P}_k(K)$ by

$$\begin{aligned} a^K\left(\Pi_{k,K}^{\nabla}v,q\right)&=a^K(v,q),\quad \forall q\in\mathbb{P}_k(K),\\ P_0\left(\Pi_{k,K}^{\nabla}v\right)&=P_0v, \end{aligned}\tag{2.1}$$

and $a^K(u,v):=\int_K \nabla u\cdot\nabla v dx$, $\forall u,v\in H^1(K)$, $v(V_i)$ represents the value of v at the i th vertex V_i, where P_0 is defined as follows:

$$P_0 v:=\frac{1}{n(K)}\sum_{i=1}^{n(K)} v(V_i), \quad k=1,$$
$$P_0 v:=\frac{1}{|K|}\int_K v dx, \quad k>1,$$

the L^2 projection $\Pi^0_{k,K}:L^2(K)\to\mathbb{P}_k(K)$ by

$$(v-\Pi^0_{k,K}v,q)_K=0, \quad \forall q\in\mathbb{P}_k(K). \tag{2.2}$$

Then, the virtual element local space is defined by

$$W^k(K)=\{\omega\in V^k(K):\int_K\left(\Pi^\nabla_{k,K}\omega\right)qdx=\int_K\omega q dx,\ q\in\mathbb{P}_k/\mathbb{P}_{k-2}(K),\ K\in T_h\},$$

the subspace $\mathbb{P}_k/\mathbb{P}_{k-2}(K)$ consists of the polynomials of k degree L^2 orthogonal to the $k-2$ degree polynomial space on K.

The global virtual element space W_h^k of order k is defined by

$$W_h^k=\{v_h\in H_0^1(\Omega):v_h|_K\in W^k(K),\ \forall K\in T_h\}.$$

The global discrete bilinear forms $a_h(\cdot,\cdot):W_h^k\times W_h^k\to R$ and $m_h(\cdot,\cdot):W_h^k\times W_h^k\to R$ are defined as

$$a_h(v,w)=\sum_{K\in T_h}a_h^K(v,w), \quad \forall v,w\in W_h^k,$$
$$m_h(v,w)=\sum_{K\in T_h}m_h^K(v,w), \quad \forall v,w\in W_h^k,$$

$a_h^K(\cdot,\cdot)$ and $m_h^K(\cdot,\cdot)$ are local computable discrete bilinear forms, we defined them as follows

$$a_h^K(v,w)=a^K\left(\Pi^\nabla_{k,K}v,\Pi^\nabla_{k,K}w\right)+S^K\left((I-\Pi^\nabla_{k,K})v,(I-\Pi^\nabla_{k,K})w\right),$$
$$m_h^K(v,w)=\left(\Pi^0_{k,K}v,\Pi^0_{k,K}w\right)_K+|K|S^K\left(\left(I-\Pi^0_{k,K}\right)v,\left(I-\Pi^0_{k,K}\right)w\right).$$

$S^K(\cdot,\cdot)$ are symmetric bilinear forms

$$S^K(v,w)=\sum_{i=1}^{N^K}X_i(v)X_i(w),$$

where $N^K=\dim W^k(K)$, $X_i(v)$ is the $i-th$ local degree of freedom on the element K.

Consistency property: $\forall p\in\mathbb{P}_k(K)$ and $v\in W^k(K)$

$$a_h^K(p,v)=a^K(p,v) \quad and \quad m_h^K(p,v)=(p,v)_K.$$

Stability: There exist four positive constants α_*, α^*, β_*, β^*, independent of h, $\forall v\in W^k(K)$

$$\alpha_*a^K(v,v)\le a_h^K(v,v)\le\alpha^*a^K(v,v),$$

$$\beta_*(v,v)\le m_h^K(v,v)\le\beta^*(v,v).$$

§3. Fully discrete scheme

The variational formulation of the problem (1.1) reads as

$$\begin{cases} find: u \in L^2(0,T,H^1(\Omega)), \\ m(u_t,v)+\varepsilon a(u,v)+(f(u),v)=0, \quad (X,t)\in\Omega\times J, \\ u(0)=u_0. \end{cases} \tag{3.1}$$

In order to be convenient for showing the fully discrete formulation, we set $\tau = T/N$, $t_n = n\tau$. Then we introduce the backward Euler method of (3.1),

$$\begin{cases} m_h(\frac{u_h^n - u_h^{n-1}}{\tau}, v_h)+\varepsilon a_h(u_h^n, v_h)+(f_h(u_h^n), v_h)=0, \quad \forall v_h \in W_h^k, \\ u_h^0 = I_h u^0, \end{cases} \tag{3.2}$$

where I_h is the interpolation operator in W_h^k. Similar to [1], for computation of nonlinear force function, on each element K, we define $f_h(u_h^n,t)$ as follows:

$$f_h(u_h^n,t)|_K := \Pi_{k,K}^0 \left(f\left(\Pi_{k,K}^0 u_h^n, t\right)\right) \text{ on each } K\in T_h, \text{ for a.e. } t\in[0,T].$$

One needs to solve a nonlinear system at each time step and this would be a tedious job. The possible remedy is to linearize the right-hand side, that is, replace $(f_h(u_h^n),v_h)$ by $(f_h(u_h^{n-1}),v_h)$ in Equation (3.2)

$$\begin{cases} m_h(\frac{u_h^n - u_h^{n-1}}{\tau}, v_h)+\varepsilon a_h(u_h^n, v_h)+(f_h(u_h^{n-1}), v_h)=0, \quad \forall v_h \in W_h^k, \\ u_h^0 = I_h u^0. \end{cases} \tag{3.3}$$

From [9], we know this scheme is stable.

Lemma 3.1. *[20] The energy projection operator $P^h: H_0^1(\Omega)\to W_h^k$ defined as*

$$a_h\left(P^h u, v_h\right) = a(u,v_h), \quad \forall v_h \in W_h^k.$$

$\forall u \in H^{k+1}(\Omega)\cap H_0^1(\Omega)$, *it holds that*

$$|u-P^h u|_1 \le Ch^k |u|_{k+1}, \tag{3.4}$$

$$||u-P^h u||_0 \le Ch^{k+1}|u|_{k+1}. \tag{3.5}$$

§4. Error estimation of the Allen-Cahn equation

4.1. Error estimation of L^2 norm

Theorem 4.1. *Let $u^n = u(\cdot,t_n)$ be the solution of problem(3.1)and let $u_h(\cdot,t_n)=u_h^n$ be the solution of problem (3.3). Then there exists a positive constant C independent of h, for all $t\in[0,T]$ we have*

$$\| u_h^n - u(\cdot,t_n)\|_0$$

$$\leq C\|u_h^0-u_0\|_0+C\tau\left(\|u_{tt}\|_{L^1(0,t_n;L^2(\Omega))}+\|u_t\|_{L^\infty(0,t_n;L^2(\Omega))}\right)$$
$$+Ch^{k+1}\left(|u_0|_{k+1}+|u_t|_{L^1(0,t;H^{k+1}(\Omega))}+\max_{1\leq j\leq n}|u(t_j)|_{k+1}+\max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}\right).$$

Proof. We set

$$u_h^n-u(\cdot,t_n)=[u_h^n-P^hu(\cdot,t_n)]+[P^hu(\cdot,t_n)-u(\cdot,t_n)]=A^n+B^n. \tag{4.1}$$

Utilizing (3.5), we obtain

$$\begin{aligned}\|B^n\|_0&=\|P^hu(\cdot,t_n)-u(\cdot,t_n)\|_0\leq Ch^{k+1}|u(\cdot,t_n)|_{k+1}\\&\leq Ch^{k+1}(|u(\cdot,0)|_{k+1}+\int_0^{t^n}|u_t(\cdot,s)|_{k+1}dx)\\&=Ch^{k+1}(|u_0|_{k+1}+|u_t|_{L^1(0,t;H^{k+1}(\Omega))}).\end{aligned} \tag{4.2}$$

It is easy to observe that

$$\begin{aligned}&m_h(\frac{A^n-A^{n-1}}{\tau},v_h)+\varepsilon a_h(A^n,v_h)\\&=m_h(\frac{u_h^n-P^hu(\cdot,t_n)-u_h^{n-1}+P^hu(\cdot,t_{n-1})}{\tau},v_h)+\varepsilon a_h(u_h^n-P^hu(\cdot,t_n),v_h)\\&=m_h(\frac{u_h^n-u_h^{n-1}}{\tau},v_h)-m_h(\frac{P^hu(\cdot,t_n)-P^hu(\cdot,t_{n-1})}{\tau},v_h)+\varepsilon a_h(u_h^n,v_h)-\varepsilon a_h(P^hu(\cdot,t_n),v_h)\\&=[m_h(\frac{u_h^n-u_h^{n-1}}{\tau},v_h)+\varepsilon a_h(u_h^n,v_h)]-m_h(\frac{P^hu(\cdot,t_n)-P^hu(\cdot,t_{n-1})}{\tau},v_h)-\varepsilon a(u(\cdot,t_n),v_h)\\&=-(f_h(u_h^{n-1},t_{n-1}),v_h)-m_h(\frac{P^hu(\cdot,t_n)-P^hu(\cdot,t_{n-1})}{\tau},v_h)+(f(u^n,t_n),v_h)+(u_t(\cdot,t_n),v_h)\\&=(f(u^n,t_n)-f_h(u_h^{n-1},t_{n-1}),v_h)+(u_t(\cdot,t_n),v_h)-m_h(\frac{P^hu(\cdot,t_n)-P^hu(\cdot,t_{n-1})}{\tau},v_h)\\&=:\mathcal{I}(v_h)+\mathcal{II}(v_h).\end{aligned} \tag{4.3}$$

The bound for $\mathcal{I}(v_h)$ is followed using Lipschitz continuity of f with respect to u and standard approximation property of L^2 projection operator, references [1],

$$\begin{aligned}\mathcal{I}(v_h)&\leq C\left(\|u_h^{n-1}-u(\cdot,t_n)\|_0+h^{k+1}|f(u(t_n),t_n)|_{k+1}+h^{k+1}|u(\cdot,t_n)|_{k+1}\right)\|v_h\|_0\\&=:C\alpha^n\|v_h\|_0.\end{aligned} \tag{4.4}$$

The second term $\mathcal{II}(v_h)$ can be estimated by the consistency and stability of the bilinear form

$$\begin{aligned}\mathcal{II}(v_h)=&\sum_{k\in T_h}(u_t(\cdot,t_n),v_h)_K-m_h^K(\frac{P^hu(\cdot,t_n)-P^hu(\cdot,t_{n-1})}{\tau},v_h)\\=&\sum_{k\in T_h}[(u_t(\cdot,t_n)-\frac{u(\cdot,t_n)-u(\cdot,t_{n-1})}{\tau},v_h)_K\\&+(\frac{u(\cdot,t_n)-u(\cdot,t_{n-1})}{\tau}-\frac{\Pi_{k,K}^0(u(\cdot,t_n)-u(\cdot,t_{n-1}))}{\tau},v_h)_K\\&+m_h^K(\frac{\Pi_{k,K}^0(u(\cdot,t_n)-u(\cdot,t_{n-1}))}{\tau}-\frac{P^hu(\cdot,t_n)-P^hu(\cdot,t_{n-1})}{\tau},v_h)]\end{aligned}$$

$$
\begin{aligned}
&\le C\|u_t(\cdot,t_n)-\frac{u(\cdot,t_n)-u(\cdot,t_{n-1})}{\tau}\|_0 \\
&\quad+\frac{1}{\tau}\|u(\cdot,t_n)-u(\cdot,t_{n-1})-\Pi^0_{k,K}(u(\cdot,t_n)-u(\cdot,t_{n-1}))\|_0 \\
&\quad+\frac{1}{\tau}\|\Pi^0_{k,K}(u(\cdot,t_n)-u(\cdot,t_{n-1}))-P^h(u(\cdot,t_n)-u(\cdot,t_{n-1}))\|_0\|v_h\|_0 \\
&\le \frac{c}{\tau}[\|\tau u_t(\cdot,t_n)-(u(\cdot,t_n)-u(\cdot,t_{n-1}))\|_0+h^{k+1}|u(\cdot,t_n)-u(\cdot,t_{n-1})|_{k+1}\|v_h\|_0 \\
&=: \frac{c}{\tau}(\beta^n+\gamma^n)\|v_h\|_0.
\end{aligned} \tag{4.5}
$$

Recalling the inequality (4.3) and setting $v_h=A^n$, we have

$$
\begin{aligned}
m_h(\frac{A^n-A^{n-1}}{\tau},A^n)+\varepsilon a_h(A^n,A^n)&=\mathcal{I}(A^n)+\mathcal{II}(A^n) \\
&\le \frac{C}{\tau}(\tau\alpha^n+\beta^n+\gamma^n)\|A^n\|_0.
\end{aligned} \tag{4.6}
$$

As $a_h(A^n,A^n)\ge 0$, we have

$$
m_h(A^n,A^n)\le C(\tau\alpha^n+\beta^n+\gamma^n)\|A^n\|_0+m_h(A^{n-1},A^n).
$$

Then

$$
\|A^n\|_0^2=m_h(A^n,A^n)\le m_h(A^{n-1},A^n)+C(\tau\alpha^n+\beta^n+\gamma^n)\|A^n\|_0.
$$

We can obtain

$$
\|A^n\|_0\le C\|A^0\|_0+C\sum_{j=1}^{n}(\tau\alpha^j+\beta^j+\gamma^j). \tag{4.7}
$$

By using (3.5), we obtain

$$
\begin{aligned}
\|A^0\|_0&=\|u_h(\cdot,0)-P^h u(\cdot,0)\|_0 \\
&=\|u_h(\cdot,0)-u_0+u_0-P^h u(\cdot,0)\|_0 \\
&\le C(\|u_h^0-u_0\|+h^{k+1}|u_0|_{k+1}).
\end{aligned} \tag{4.8}
$$

Further, we have

$$
\begin{aligned}
\tau\sum_{j=1}^{n}\alpha^j\le{}& Ch^{k+1}\max_{1\le j\le n}|f(u(t_j),t_j)|_{k+1}+Ch^{k+1}\max_{1\le j\le n}|u(t_j)|_{k+1}+C\tau\|u_t\|_{L^\infty(0,t_n,L^2(\Omega))} \\
&+Ch^{k+1}(|u_0|_{k+1}+|u_t|_{L^1(0,t;H^{k+1}(\Omega))})+C\|u_h^0-u_0\|_0.
\end{aligned} \tag{4.9}
$$

For β^j, we have

$$
\begin{aligned}
\beta^j&=\|u(\cdot,t_j)-u(\cdot,t_{j-1})-\tau u_t(\cdot,t_j)\|_0 \\
&=\|-\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_{tt}(\cdot,s)ds\|_0 \\
&\le \int_{t_{j-1}}^{t_j}\|u_{tt}(\cdot,s)\|_0(s-t_{j-1})ds\le \tau\int_{t_{j-1}}^{t_j}\|u_{tt}(\cdot,s)\|_0 ds,
\end{aligned}
$$

$$\sum_{j=1}^{n}\beta^j \leq \tau\int_0^{t_n}\|u_{tt}(\cdot,s)\|_0 ds = \tau\|u_{tt}\|_{L^1(0,t_n;L^2(\Omega))}. \tag{4.10}$$

For γ^j, it holds

$$\begin{aligned}\gamma^j &= h^{k+1}|u(\cdot,t_n)-u(\cdot,t_{n-1})|_{k+1}\\ &= h^{k+1}|-\int_{t_{j-1}}^{t_j} u_t(\cdot,s)ds|_{k+1}\\ &\leq h^{k+1}|\int_{t_{j-1}}^{t_j}|u_t(\cdot,s)|_{k+1}ds,\end{aligned}$$

$$\sum_{j=1}^{n}\gamma^j \leq h^{k+1}\int_0^{t_n}|u_t(\cdot,s)|_{k+1}ds = |u_t|_{L^1(0,t_n;H^{k+1}(\Omega))}. \tag{4.11}$$

Combining (4.7)-(4.11) and (4.2), we complete the proof. □

4.2. Error estimation of H^1 norm

Theorem 4.2. *Let $u^n = u(\cdot,t_n)$ be the solution of problem(3.1)and let $u_h(\cdot,t_n) = u_h^n$ be the solution of problem (3.3). Then there exists a positive constant C independent of h, for all $t\in[0,T]$ we have*

$$\begin{aligned}&|u_h^n - u(\cdot,t_n)|_{1,h}\\ \leq & C|u_h^0-u_0|_1 + \frac{C}{\sqrt{\varepsilon}}\Big(\tau\|u_{tt}\|_{L^2(0,t_n;L^2(\Omega))}+\tau\|u_t\|_{L^\infty(0,t_n;L^2(\Omega))}\Big)\\ &+\frac{C}{\sqrt{\varepsilon}}h^k(|u_0|_{k+1}+\|u_t\|_{L^1(0,t_n;H^{k+1}(\Omega))}+h\|u_t\|_{L^2(0,t_n;H^{k+1}(\Omega))}\\ &+h\max_{1\leq j\leq n}|u(t_j)|_{k+1}+h\max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}).\end{aligned}$$

Proof. Similar to the proof of Theorem 4.1, substituting (4.4) and (4.5) into (4.3)

$$m_h(\frac{A^n-A^{n-1}}{\tau},v_h)+\varepsilon a_h(A^n,v_h) = \mathcal{I}(v_h)+\mathcal{II}(v_h) \leq \frac{c}{\tau}(\tau\alpha^n+\beta^n+\gamma^n)\|v_h\|_0,$$

and set $v_h = \frac{A^n-A^{n-1}}{\tau}$, we can get

$$\begin{aligned}&m_h(\frac{A^n-A^{n-1}}{\tau},\frac{A^n-A^{n-1}}{\tau})+\varepsilon a_h(A^n,\frac{A^n-A^{n-1}}{\tau})\\ \leq & \frac{C}{\tau}(\tau\alpha^n+\beta^n+\gamma^n)\|\frac{A^n-A^{n-1}}{\tau}\|_0.\end{aligned} \tag{4.12}$$

Observing

$$\begin{aligned}a_h(A^n,\frac{A^n-A^{n-1}}{\tau}) &= \frac{1}{\tau}(a_h(A^n,A^n)-a_h(A^n,A^{n-1}))\\ &= \frac{1}{\tau}(\|A^n\|_1^2 - a_n(A^n,A^{n-1}))\\ &\geq \frac{1}{\tau}(\|A^n\|_1^2-\|A^n\|_1\|A^{n-1}\|_1)\end{aligned} \tag{4.13}$$

$$\begin{aligned}&\geq \frac{1}{\tau}(\|A^n\|_1^2-\frac{1}{2}\|A^n\|_1^2-\frac{1}{2}\|A^{n-1}\|_1^2)\\&=\frac{1}{2\tau}(\|A^n\|_1^2-\|A^{n-1}\|_1^2),\end{aligned}$$

and the stability property of the bilinear form $m_h(\cdot,\cdot)$, enable us to write

$$m_h(\frac{A^n-A^{n-1}}{\tau},\frac{A^n-A^{n-1}}{\tau})\geq \beta_*\|\frac{A^n-A^{n-1}}{\tau}\|_0^2. \tag{4.14}$$

Then, we use (4.13) and (4.14) to obtain

$$m_h(\frac{A^n-A^{n-1}}{\tau},\frac{A^n-A^{n-1}}{\tau})+\varepsilon a_h(A^n,\frac{A^n-A^{n-1}}{\tau})\geq \beta_*\|\frac{A^n-A^{n-1}}{\tau}\|_0^2+\frac{\varepsilon}{2\tau}(\|A^n\|_1^2-\|A^{n-1}\|_1^2),$$

using (4.12), we get

$$\begin{aligned}&\beta_*\|\frac{A^n-A^{n-1}}{\tau}\|_0^2+\frac{\varepsilon}{2\tau}(\|A^n\|_1^2-\|A^{n-1}\|_1^2)\\&\leq \frac{C}{\tau}(\tau\alpha^n+\beta^n+\gamma^n)\|\frac{A^n-A^{n-1}}{\tau}\|_0\\&\leq \frac{C}{\tau^2}(\tau\alpha^n+\beta^n+\gamma^n)^2+\|\sqrt{\beta_*}\frac{A^n-A^{n-1}}{\tau}\|_0^2\\&\leq \frac{C}{\tau^2}(\tau^2(\alpha^n)^2+(\beta^n)^2+(\gamma^n)^2)+\beta_*\|\frac{A^n-A^{n-1}}{\tau}\|_0^2,\end{aligned}$$

which leads to

$$\|A^n\|_1^2\leq\|A^0\|_1^2+\frac{C}{\tau\varepsilon}\sum_{j=1}^{n}(\tau^2(\alpha^j)^2+(\beta^j)^2+(\gamma^j)^2),$$

together with the stability of $a_h(\cdot,\cdot)$ implies

$$|A^n|_1^2\leq|A^0|_1^2+\frac{C}{\tau\varepsilon}\sum_{j=1}^{n}(\tau^2(\alpha^j)^2+(\beta^j)^2+(\gamma^j)^2). \tag{4.15}$$

Similar to (4.8)-(4.11), we have

$$\begin{aligned}|A^0|_1&=|u(\cdot.0)-u_h(\cdot.0)|_1+|u(\cdot.0)-P^hu(\cdot.0)|_1\\&\leq|u_h^0-u_0|_1+Ch^k|u_0|_{k+1},\end{aligned} \tag{4.16}$$

$$\begin{aligned}\tau^2\sum_{j=1}^{n}\left(\alpha^j\right)^2\leq{}&\tau h^{2(k+1)}\max_{1\leq j\leq n}|f\left(u\left(t_j\right),t_j\right)|_{k+1}^2+\tau h^{2(k+1)}\max_{1\leq j\leq n}|u\left(t_j\right)|_{k+1}^2\\&+\tau^3\|u_t\|_{L^\infty(0,t_n,L^2(\Omega))}^2+\tau h^{2k}|u_0|_{k+1}^2+\tau h^{2k}|u_t|_{L^1(0,t;H^{k+1}(\Omega))}^2,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\sum_{j=1}^{n}(\beta^j)^2&\leq\tau^3\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|u_{tt}(\cdot,s)\|^2ds\\&=\tau^3\int_0^{t_n}\|u_{tt}(\cdot,s)\|^2ds=\tau^3\|u_{tt}\|_{L^2(0,t_n;L^2(\Omega))}^2,\end{aligned} \tag{4.18}$$

$$\begin{aligned}\sum_{j=1}^{n}(\gamma^j)^2 &\leq \tau h^{2(k+1)}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}|u_t(\cdot,s)|_{k+1}^2 ds\\ &= \tau^{2(k+1)}\int_0^{t_n}|u_t(\cdot,s)|_{k+1}^2 ds = \tau h^{2(k+1)}\|u_t\|^2_{L^2(0,t_n;H^{k+1}(\Omega))},\end{aligned} \tag{4.19}$$

we substitute (4.16)-(4.19) into (4.15) to obtain

$$\begin{aligned}|A^n|_1 \leq& |u_h^0-u_0|_1+\frac{C}{\sqrt{\varepsilon}}(h^{k+1}\max_{1\leq j\leq n}|f(u(t_j),t_j)|_{k+1}\\ &+h^{k+1}\max_{1\leq j\leq n}|u(t_j)|_{k+1}+\tau\|u_t\|_{L^\infty(0,t_n,L^2(\Omega))})\\ &+\frac{C}{\sqrt{\varepsilon}}h^k(|u_0|_{k+1}+|u_t|_{L^1(0,t;H^{k+1}(\Omega))})+\tau\|u_{tt}\|_{L^2(0,t_n;L^2(\Omega))}+\tau h^{k+1}\|u_t\|_{L^2(0,t_n;H^{k+1}(\Omega))}).\end{aligned} \tag{4.20}$$

By (4.1), we have

$$\begin{aligned}|u_h^n-u(\cdot,t_n)|_1 &= |[u_h^n-P^h u(\cdot,t_n)]+[P^h u(\cdot,t_n)-u(\cdot,t_n)]|_1\\ &\leq |A^n|_1+|B^n|_1.\end{aligned} \tag{4.21}$$

The estimate of B^n is given by

$$|B^n|_1 \leq Ch^k(|u_0|_{k+1}+\|u_t\|_{L^2(0,t_n;H^{k+1}(\Omega))}). \tag{4.22}$$

Combining (4.20)-(4.22) completes the proof of the theorem. □

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