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Characters of the Solutions to a Generalized Nonlinear Max-type Difference Equation

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Abstract: This paper is concerned with the boundedness and asymptotic behavior of positive solutions for a generalized difference equation arising from automatic control theory. The main results improve and extend the ones in the previous works to a large extent. One in particular is that Rouche's theorem is available to prove the convergence of solutions.

Key words: nonlinear; max-type; boundedness; asymptotic behavior

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§1. Introduction

As a discrete analogue of differential or delay differential equation, difference equation arises naturally in various scientific branches[1-5]. Recently the so-called max-type difference equation has been receiving great attention(see [6-12]). Many particular works have been previously finished, mostly by Stević and his collaborators. One can refer to [7, 8, 12-14, 17-18], as well as the references therein. The present problem is about the boundedness and behavior of positive solutions for a generalized max-type difference equation from automatic control theory [2, 15-16]. As we all know, this is an important issue since it is a basic precondition for the establishment of stability or periodicity of all solutions to the equation.

As a generalized equation and a continuation of previous work, in [12] they proposed the necessity of investigating a new research direction

$$x_{n+1} = \max\{A, \frac{x_n^p}{x_{n-k}^q x_{n-m}^r}\}, \quad n \in \mathbb{N}_{\not{\ }}. \tag{1.1}$$

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Indeed the behavior of Eq (1.1) is very complicated. Here, as a primary extension, we study the positive solutions to a special case of Eq (1.1) as follow

$$x_{n+1} = \max\{A, \frac{x_n^p}{x_{n-1}^q x_{n-k}^r}\}, \quad n \in \mathbb{N}_{\not k}, \tag{1.2}$$

where $k \geq 2$ and $A,\ p,\ q,\ r$ are positive numbers.

§2. Boundedness of Solutions

In this section, we investigate the boundedness character of the positive solutions to Eq (1.2).

Theorem 2.1 Assume that $p \leq 1+q+r$. If $1 < q < p^2 < 4q$ or $\frac{p+\sqrt{p^2-4q}}{2} > 1$, then all positive solutions to Eq (1.2) are bounded.

Proof Remark Eq (1.2) as the following form

$$x_{n+1} = \max\{A, \frac{x_n^p}{x_{n-1}^q x_{n-k}^r}\} = \max\{A, \frac{x_n^p}{x_{n-1}^q x_{n-2}^0 \cdots x_{n-k}^r}\}. \tag{2.1}$$

By the definition of x_n produced by Eq (1.2), we can deduce

$$(2.1) = \max\{A, \frac{A^p}{x_{n-1}^q x_{n-2}^0 \ldots x_{n-k}^r}, \frac{x_{n-1}^{p^2-q}}{x_{n-2}^{qp} x_{n-3}^0 \cdots x_{n-k}^r x_{n-k-1}^{rp}}\}, \quad n > k+1, \tag{2.2}$$

we can see that if $p^2 < q$, then from (2.2) it yields

$$x_{n+1} = \max\{A, \frac{A^p}{A^{q+r}}, \frac{1}{A^{r+q+pq+rq-p^2}}\}, \tag{2.3}$$

which implies the boundedness of solutions in this case.

Now assume that $p^2 > q$. From (2.2), by continuing with iterations, we have

$$\begin{aligned}
x_{n+1} &= \max\{A, \frac{A^p}{x_{n-1}^q x_{n-k}^r}, \frac{x_{n-1}^{p^2-q}}{x_{n-2}^{qp} x_{n-k}^r x_{n-k-1}^{pr}}\} \\
&= \max\{A, \{\frac{A}{x_{n-1}^{\frac{q}{p}} x_{n-k}^{\frac{r}{p}}}\}^p, \{\frac{x_{n-1}^{p-\frac{q}{p}}}{x_{n-2}^q x_{n-k}^{\frac{r}{p}} x_{n-k-1}^r}\}^p\} \\
&= \max\{A, \{\frac{A}{x_{n-1}^{\frac{q}{p}} x_{n-k}^{\frac{r}{p}}}\}^p, ((\frac{x_{n-1}}{x_{n-2}^{q/(p-\frac{q}{p})} x_{n-k}^{\frac{r}{p}/(p-\frac{q}{p})} x_{n-k-1}^{r/(p-\frac{q}{p})}})^{p-\frac{q}{p}})^p\} \\
&= \cdots \\
&= \max\{A, (\frac{A}{x_{n-1}^{\frac{q}{p}} x_{n-k}^{\frac{r}{p}}}, (\cdots, (\frac{x_{n-l}}{x_{n-l-1}^{q/p-a_l} x_{n-k}^{r/p(p-a_1)\ldots(p-a_l)} \ldots x_{n-l-k}^{r/p-a_l}})^{p-a_l} \cdots)^{p-\frac{q}{p}})^p\} \\
&= \max\{A, (\cdots, (\frac{A}{x_{n-l-1}^{q/p-a_l} \cdots x_{n-l-k}^{r/p-a_l}}, \frac{x_{n-l-1}^{(p-a_{l+1})}}{x_{n-l-2}^q \ldots x_{n-l-k}^{r/p-a_l} x_{n-l-k-1}^r})^{p-a_l} \cdots)^{p-\frac{q}{p}})^p\},
\end{aligned} \tag{2.4}$$

here $a_{l+1} = q/(p-a_l)$ with $a_0 = 0$ and $a_1 = q/p$.

Since $p^2 > q$, we have

$$0 = a_0 < a_1 = q/p < a_2 = q/(p - q/p). \tag{2.5}$$

By mathematical induction, it is easy to deduce

$$a_l < a_{l+1}, \tag{2.6}$$

which implies that the sequence $\{a_l\}$ is strictly increasing.

In what follows, we prove the sequence $\{a_l\}$ is bounded. Remark that the polynomial

$$P(\lambda) = \lambda(p - \lambda) - q. \tag{2.7}$$

If the condition $1 < q < p^2 < 4q$ or $\frac{p+\sqrt{p^2-4q}}{2} > 1$ holds, which implies that the modulus of complex or real root is greater than one, then from this it follows that $\{a_l\}$ is unbounded. Hence there is the least $l_0 \in \mathbb{N}$ such that $a_{l_0} < p$ and $a_{l_0+1} \geq p$. Using this fact and (2.4) with $l = l_0 + 1$, it follows that

$$\begin{aligned} x_{n+1} &= \max\{A, (\cdots, (\frac{A}{x_{n-l-1}^{q/p-a_l} \cdots x_{n-l-k}^{r/p-a_l}}, \frac{x_{n-l-1}^{(p-a_{l+1})}}{x_{n-l-2}^{q} \cdots x_{n-l-k}^{r/p-a_l} x_{n-l-k-1}^{r}})^{p-a_l} \cdots)^{p-\frac{q}{p}})^{p}\} \\ &= \max\{A, (\cdots, (\frac{A}{A^{q/p-a_{l_0}} \cdots A^{r/p-a_{l_0}}}, \frac{1}{A^{(a_{l_0+1}-p)} A^{q} \cdots A^{r/p-a_{l_0}} A^{r}})^{p-a_{l_0}} \cdots)^{p-\frac{q}{p}})^{p}\}, \end{aligned} \tag{2.8}$$

from which the boundedness of the sequence x_n yields, as desired.

§3. Behavior of Solutions

In this section we consider the asymptotic behavior of the positive solution of (1.2). Without loss of generality, first considering the difference equation

$$x_{n+1} = \max\{1, \frac{x_n^p}{x_{n-1}^q x_{n-k}^r}\}, \quad n \in \mathbb{N}_{\not k}, \tag{3.1}$$

with positive initial data $x_0, x_{-1}, \cdots, x_{-k}$ and $A = 1$.

Obviously, by the change $x_n = B^{y_n}$ with $B > 1$, Eq (3.1) is transformed into the following difference equation

$$y_{n+1} = \max\{0, py_n - qy_{n-1} - ry_{n-k}\}. \tag{3.2}$$

which implies that $y_n \geq 0$ for $n \in \mathbb{N}$, $n > k$. From here we can obtain a set of initial data $y_0, y_{-1}, \cdots$, and y_{-k}. Furthermore, by iterating this step, we can get a set of positive data $y_{k+1}, y_k, \cdots$, and y_1. To this end, we have the following result in this case.

Theorem 3.1 Let $\{x_n\}$ be a solution of Eq (3.1). Assume $p \leq 1$ holds. Then $\{x_n\}$ converges to $x^* = 1$.

Proof It is clear that the new set of data $y_{k+1}, y_k, \cdots$, and y_1 produced by iteration of Eq (3.2) are nonnegative.

If $py_{k+1} - qy_k - ry_1 \leq 0$, then we have $y_{k+2} = 0$. Furtherly, $y_{k+3} = \max\{0, py_{k+2} - qy_{k+1} - ry_3\} = \max\{0, -qy_{k+1} - ry_3\} = 0$ and $y_{k+4} = \max\{0, py_{k+3} - qy_{k+2} - ry_4\} = \max\{0, -ry_4\} = 0$. Iterate the procedure, we have $y_n = 0$ for $n > k+1$.

Otherwise, if $py_{k+1} - qy_k - ry_1 > 0$, at this moment, we have

$$y_{k+2} = \max\{0, py_{k+1} - qy_k - ry_1\} = py_{k+1} - qy_k - ry_1. \tag{3.3}$$

Namely,

$$y_{k+2} + qy_k + ry_1 = py_{k+1}. \tag{3.4}$$

Therefore, $y_{k+2} \leq py_{k+1}$. Since $p \leq 1$, it follows $y_{k+2} \leq y_{k+1}$. Iterating this step, if there exists a $N > k+2$ such that $py_N - qy_{N-1} - ry_{N-k} \leq 0$, then it will become the first case which implies the convergence of solutions. If not like this case, for any $n > k+1$, $py_n - qy_n - ry_{n-k} > 0$. From this it follows $y_{n+1} \leq py_n \leq y_n$ with the assumption $p \leq 1$. Indeed we obtain a monotone decreasing sequence $\{y_n\}$ with $n > k+1$. Due to the positivity of $\{y_n\}$, we can deduce there exists a limit denoted by y^* such that $\lim\limits_{n\to\infty} y_n = y^*$. Since $1 + q + r > 1 \geq p$, it follows $y^* = 0$ from the equality (3.4). All arguments above with the help of the change $x_n = B^{y_n}$ with $B > 1$ conclude our proof.

Next we consider the case of $A > 1$. The change $x_n = A^{y_n}$ carries Eq (3.1) into the difference equation

$$y_{n+1} = \max\{1, py_n - qy_{n-1} - ry_{n-k}\}.$$

which implies that $y_{n+1} \geq 1$ for $n > k$. Moreover, we can rewrite it to the following form

$$y_{n+1} - 1 = \max\{0, py_n - qy_{n-1} - ry_{n-k} - 1\},$$

which is almost the same sequence with Eq (3.2). Therefore we have the following corollary by the similar arguments in Theorem 3.1.

Corollary 3.1 Let $\{x_n\}$ be a solution of Eq (1.2). Assume $p \leq 1$, then $\{x_n\}$ converges to $x^* = A$. In addition, if $p - q - r > 1$, and initial data $x_{-k}, \cdots, x_{-1}, x_0$ are not greater than x_0, and at least one data is greater than one.the solution $\{x_n\}$ is a divergent sequence.

Finally we consider the case of $A < 1$. At this moment, the same change $x_n = A^{y_n}$ yields the following equation

$$y_{n+1} = \min\{1, py_n - qy_{n-1} - ry_{n-k}\}, \tag{3.5}$$

which implies $0 \leq y_{n+1} \leq 1$ for $n > k$. By Rouche's theorem, we can obtain the following result.

Theorem 3.2 Let $\{x_n\}$ be a solution of Eq (1.2). If one of the following assumptions (i) $p - q - r \geq 1$; (ii) $\max\{p, q, r\} \leq \frac{1}{3}$ holds . Then $\{x_n\}$ converges to $x^* = A$.

Proof If there exists $N \in \mathbb{N}$, $N > k$ such that $py_N - qy_{N-1} - ry_{N-k} \geq 1$, then from (3.5) we have $y_{N+1} = 1$. Moreover, it can be derived $p - q - r \leq y_{N+2} \leq p$. According the hypothesis condition, (i) and Eq (3.5), it follows $1 \leq p - q - r \leq y_{N+2} \leq p$ which implies $y_{N+2} = 1$. By iterating this process, we have $y_n = 1$ for $n > N$, which yields the convergence of solution $\{x_n\}$ with the help of the change above.

Otherwise, for any $n \in \mathbb{N}$, if $py_n - qy_{n-1} - ry_{n-k} < 1$, then we have

$$y_{n+1} = py_n - qy_{n-1} - ry_{n-k}. \tag{3.6}$$

The characteristic polynomial associated with Eq (3.6) is

$$P(\lambda) = \lambda^{k+1} - p\lambda^k + q\lambda^{k-1} + r. \tag{3.7}$$

Let $f(z) = z^{k+1}$ and $g(z) = pz^k - qz^{k-1} - r$. Note that the condition $\max\{p, q, r\} \le \frac{1}{3}$, we have that on the unit circle $|z| = 1$,

$$\begin{aligned} |g(z)| &\le |pz^k - qz^{k-1} - r| \\ &\le \max\{p, q, r\}(|z|^k + |z|^{k-1} + 1) \\ &\le 3\max\{p, q, r\} \le 1 = |z|^{k+1} = |f(z)|. \end{aligned} \tag{3.8}$$

By Rouche's theorem it follows that the polynomials $f(z)$ and $f(z)-g(z)$ have the same number of zeroes in the unit disk $|z| < 1$. Since $f(z)$ has $k+1$ zeroes in the unit disk, it follows that the polynomial $f(z) - g(z) = z^{k+1} - pz^k + qz^{k-1} + r$ has also $k+1$ zeroes in the unit disk.

Let $\lambda_1, \cdots, \lambda_u (u \le k-1)$ be different zeroes of the polynomial $f - g$, with the multiplicities $v_j,\ j = 1, \cdots, u$. Then

$$y_{n+1} = \sum_{j=1}^{u} F_j(n)\lambda_j^n, \tag{3.9}$$

for some polynomials $F_j,\ j = 1, \cdots, u$. Clearly y_{n+1} converges, which along with the change $x_n = A^{y_n}$ implies that the sequences $\{x_n\}$ converge too, denoted by x^*. From Eq (3.6) and the hypothesis condition we have $x^* = A$.

Remark The hypothesis conditions (ii) in Theorem 3 can be further rewritten into much precise and exact ones. Namely, (iii) $\max\{q, r\} \le \frac{1-p}{2}$ for $p \le 1$ or (iv) $\max\{p, r\} \le \frac{1-q}{2}$ for $q \le 1$ or (v) $\max\{p, q\} \le \frac{1-r}{2}$ for $r \le 1$. Indeed if let $f(z) = z^{k+1} - pz^k$ and $g(z) = qz^{k-1} + r$ or $f(z) = z^{k+1} + qz^{k-1}$ and $g(z) = pz^k - r$ or $f(z) = z^{k+1} + r$ and $g(z) = pz^k - qz^{k-1}$, respectively. The same analysis approach is applied to the assumption (iii), (iv) and (v), which follows our claim.

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