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Strongly Ding projective modules with respect to a semidualizing module

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Abstract: This paper is a study of strongly Ding projective modules with respect to a semidualizing module. The class of strongly Ding flat modules with respect to a semidualizing module is also investigated, and the relationship between strongly Ding projective modules and strongly Ding flat modules with respect to a semidualizing module is characterized. Some well-known results on strongly Ding projective modules, n-strongly Ding projective modules and strongly D_C-projective modules are generalized and unified.
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§1. Introduction

Throughout, unless otherwise indicated, all rings are commutative rings with identity and all modules are unitary modules. Auslander and Bridger[1] introduced the G-dimension for finitely generated modules. Enochs and Jenda defined and studied the Gorenstein projective, Gorenstein injective[8] and Gorenstein flat modules [10] and developed Gorenstein homological algebra [9] on modules over a ring R. Various generalizations of these modules are given over specific rings (see, e.g., [12-14, 17-18, 23, 27, 31-32]). As special cases of Gorenstein projective and Gorenstein injective modules, strongly Gorenstein flat and Gorenstein FP-injective modules were studied in [6-7], and later in [11] under different names — the Ding projective and Ding injective modules. It was shown in [6] that the class of Ding projective modules over coherent rings are indeed stronger than Gorenstein flat modules. This class of modules can gives some

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new interesting characterizations of rings. For example, it was proved in [6, Proposition 2.15] that a ring R is right perfect if and only if every flat right R-module is Ding projective.

Recently, Ding projective dimensions and strongly Ding projective modules were further investigated in [30]. More generally, Ding projective modules and Ding projective dimensions with respect to a semidualizing module were studied in [29] and [31]. In this paper, we study the concept of strongly Ding projective modules with respect to a semidualizing module. The properties of strongly D_C-flat modules under change of rings are also investigated. The paper is organized as follows. In section 2, we give some notions and definitions which we need in the later sections. Section 3 is a study of strongly Ding projective modules with respect to a semidualizing module. The concept of strongly D_C-projective modules is introduced and some characterizations of this class of modules are given. Strongly Ding projective modules are a particular case of strongly D_C-projective modules when $C = R$. In Section 4, we study the properties of strongly Ding flat modules with respect to a semidualizing module. It is proved that if M is a strongly D_C-flat module, then M^+ is strongly D_C-injective. Some results on strongly Ding flat modules are obtained as corollaries of these results.

§2. Preliminaries

An R-module M is called FP-injective [21] in case $\mathrm{Ext}^1_{\mathrm{R}}(P, M) = 0$ for every finite presented R-module P. A degreewise finite projective resolution of an R-module M is a projective resolution $\mathcal{P}$ of M such that each P_i in $\mathcal{P}$ is finitely generated projective. A class of modules $\mathcal{X}$ is projectively resolving if it is closed under extensions, kernels of surjections and it contains all projective modules [17].

Definition 2.1 An R-module C is semidualizing if the following conditions are satisfied:

(1) C admits a degreewise finite projective resolution,

(2) The natural homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism,

(3) $\mathrm{Ext}^{\geq 1}_R(C, C) = 0$.

Two examples of semidualizing modules are the free module of rank 1, and the dualizing (canonical) module over a Cohen-Macaulay local ring, when it exists.

Definition 2.2 Let C be a semidualizing R-module. Set

$$\mathcal{P}_C(R) = \{C \otimes_R P |\ P \text{ is } R\text{-projective}\},$$

$$\mathcal{F}_C(R) = \{C \otimes_R F |\ F \text{ is } R\text{-flat}\},$$

$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C, I) |\ I \text{ is } R\text{-injective}\},$$

$$\mathcal{FI}_C(R) = \{\mathrm{Hom}_R(C, E) |\ E \text{ is } R\text{-FP-injective}\}.$$

Modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, $\mathcal{I}_C(R)$ and $\mathcal{FI}_C(R)$ are called C-projective, C-flat, C-injective and C-FP-injective, respectively.

Clearly, if $C = R$, then $\mathcal{P}_R(R)$, $\mathcal{F}_R(R)$, $\mathcal{I}_R(R)$ and $\mathcal{FI}_R(R)$ are just the classes of ordinary *projective*, *flat*, *injective* and FP-*injective* R-modules, respectively, which are denoted by $\mathcal{F}(R), \mathcal{P}(R), \mathcal{I}(R)$ and $\mathcal{FI}(R)$, respectively, for simplicity. By [19, Proposition 5.1], $\mathcal{P}_C(R)$ is closed under direct sums and direct summands, and $\mathcal{I}_C(R)$ is closed under direct products and direct summands. Moreover, the classes $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are projectively resolving, and the class $\mathcal{I}_C(R)$ is injectively resolving by [19, Corollary 6.4].

Definition 2.3[6−7, 11] An R-module M is called Ding projective if there exists an exact sequence of projective R-modules

$$\mathcal{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M \cong \text{Coker}\ (P_1 \to P_0)$ such that $\text{Hom}_{\text{R}}(\mathcal{P}, \mathcal{F}(R))$ is exact. Ding injective module is dually defined. An R-module N is called Ding flat if there exists an exact sequence of flat R-modules

$$\mathcal{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

with $N \cong \text{Coker}\ (F_1 \to F_0)$ such that $\mathcal{FI}(R) \otimes_R \mathcal{F}$ is exact.

It is easy to see that Ding projective (resp., Ding injective) modules are Gorenstein projective (resp., Gorenstein injective). But Gorenstein injective modules over a Noetherian ring is Ding injective.

Definition 2.4 Let $\mathcal{X}$ be a class of R-modules and M an R-module. Then an $\mathcal{X}$-resolution of M is a complex of R-modules in $\mathcal{X}$ of the form

$$\mathcal{X}_\bullet = \cdots \to X_n \to \cdots \to X_1 \to X_0 \to 0$$

such that $\text{H}_0(\mathcal{X}_\bullet) \cong M$ and $\text{H}_n(\mathcal{X}_\bullet) = 0$ for all $n \geq 1$. Moreover, the exact sequence

$$\mathcal{X}_\bullet{}^+ = \cdots \to X_n \to \cdots \to X_0 \to M \to 0$$

is called the augmented $\mathcal{X}$-resolution of M associated to $\mathcal{X}_\bullet$.

Given a class $\mathcal{X}$ of R-modules and a complex $\mathbb{Y}$, we say $\mathbb{Y}$ is $\text{Hom}_R(\mathcal{X}, -)$-exact if the complex $\text{Hom}_R(X, \mathbb{Y})$ is exact for each $X \in \mathcal{X}$. Dually, the complex $\mathbb{Y}$ is $\text{Hom}_R(-, \mathcal{X})$-exact if $\text{Hom}_R(\mathbb{Y}, X)$ is exact for each $X \in \mathcal{X}$, and $\mathbb{Y}$ is $- \otimes_R \mathcal{X}$-exact if $\mathbb{Y} \otimes_R X$ is exact for each $X \in \mathcal{X}$. The $\mathcal{X}$-projective dimension of M is defined as

$$\mathcal{X}\text{-pd}_{\text{R}}(M) = \inf\{\sup\ \{n \geq 0 |\ X_n \neq 0\} | \mathcal{X}_\bullet \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

An $\mathcal{X}$-resolution $\mathcal{X}_\bullet$ of M is proper if the augmented resolution $\mathcal{X}_\bullet{}^+$ is $\text{Hom}_R(\mathcal{X}, -)$-exact. The (proper) $\mathcal{X}$-coresolution and $\mathcal{X}$-injective dimensions can be defined dually. And the $\mathcal{X}$-injective dimension of M is denoted by $\mathcal{X}\text{-id}_R(M)$.

Note 2.5 Let C be a semidualizing R-module. We use the following abbreviations

$$\mathcal{P}_C\text{-pd}_R(-) = \mathcal{P}_C(R)\text{-pd}(-),\ \mathcal{F}_C\text{-pd}_R(-) = \mathcal{F}_C(R)\text{-pd}(-)$$

Over a commutative Noetherian ring R, Foxby [14] studied two subcategories of the category of R-modules relative to a (semi)dualizing module C, which are the *Auslander class* $\mathcal{A}_C(R)$ and *Bass class* $\mathcal{B}_C(R)$. In the non-Noetherian setting, these definitions are taken from [19, 26].

Definition 2.6 The Auslander class $\mathcal{A}_C(R)$ with respect to C consists of all R-modules M satisfying: $\mathrm{Tor}^R_{\geq 1}(C,M) = 0{=}\mathrm{Ext}^{\geq 1}_R\ (C, C\otimes_R M)$, and the natural map $\mu_M : M \to \mathrm{Hom}_R(C, C\otimes_R M)$ is an isomorphism.

The Bass class of R with respect to C, denoted $\mathcal{B}_C(R)$, consists of all R-modules N satisfying: $\mathrm{Ext}^{\geq 1}_R(C,N) = 0 = \mathrm{Tor}^R_{\geq 1}(C, \mathrm{Home}_{\mathrm{R}}(C,N))$, and the natural evaluation map $\nu_N : C\otimes_R \mathrm{Hom}_{\mathrm{R}}(C,N) \to N$ is an isomorphism.

§3. Strongly D_C-projective Modules

In this section, we introduce and study strongly Ding projective modules with respect to a semidualizing module.

We begin with the following

Definition 3.1 An R-module M is said to be n-strongly Ding projective with respect to C (or simply n-strongly D_C-projective), if there exists an exact sequence

$$\mathcal{P} = 0 \to M \to C\otimes_R P_n \to \cdots \to C\otimes_R P_1 \to M \to 0$$

such that $\mathcal{P}$ is $\mathrm{Hom}_R(-,\mathcal{F}_C(R))$-exact, where each P_i is projective for all $1 \leq i \leq n$.

If $C = R$, then we call a n-strongly D_C-projective module a n-strongly Ding projective module. It is easy to see that every n-strongly Ding projective module is n-strongly Gorenstein projective [2]. We denote by n-$\mathcal{SDP}_C(R)$ the subcategory of n-strongly D_C-projective R-modules. The subcategory of n-strongly Ding projective modules is denoted by n-$\mathcal{SDP}(R)$.

Remark 3.2

(1) We call a 1-strongly D_C-projective module a strongly D_C-projective module, for simplicity. Clearly, 1-strongly D_R-projective modules are precisely the class of strongly Ding projective modules. Every strongly Ding projective module is strongly Gorenstein projective [3].

(2) Let M be an n-strongly D_C-projective module and $K_i = \mathrm{Ker}\ (C\otimes_R P_i \to C\otimes_R P_{i-1})$ with $1 \leq i \leq n$ $(C\otimes_R P_0 = M)$. It is easy to see that each K_i is n-strongly D_C-projective for all i. Similarly, we can easily check that each $H_i = \mathrm{Im}\ (C\otimes_R P_i \to C\otimes_R P_{i-1})$ is n-strongly D_C-projective for all $2 \leq i \leq n$.

Recall that an R-module M is Ding projective with respect to C (or simply D_C-projective), if there exists an exact sequence of C-projective R-modules

$$\cdots \to C\otimes_R P_1 \to C\otimes_R P_0 \to C\otimes_R P^0 \to C\otimes_R P^1 \to \cdots$$

with $M \cong \mathrm{Coker}\ (C\otimes_R P_1 \to C\otimes_R P_0)$ such that the sequence is $\mathrm{Hom}_R(-,\mathcal{F}_C(R))$-exact.

The next lemma is a corollary of [29, Theorem 2.7], which shows that the class of D_C-projective modules coincides with the D_C-projective modules defined in [29].

Lemma 3.3 An R-module M is D_C-projective if and only if there exists a $\mathrm{Hom}_{\mathrm{R}}(-,\mathcal{F}_C(R))$-exact exact sequence

$$\mathcal{P} = \cdots \to D_1 \to D_0 \to D^0 \to D^1 \to \cdots$$

such that all D_i, $D^i \in \mathcal{P}_C(R) \cup \mathcal{P}(R)$ and $M \cong$ Coker $(D_1 \to D_0)$.

The following result gives some equivalent characterizations of n-strongly D_C-projective modules.

Theorem 3.4 Let M be an R-module. Then the following are equivalent

(1) M is an n-strongly D_C-projective module.

(2) There exists an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha_{n+1}} C \otimes_R P_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} C \otimes_R P_1 \xrightarrow{\alpha_1} M \longrightarrow 0$$

with each P_i projective such that $\mathrm{Ext}_R^{\geq 1}(M, \mathcal{F}_C(R)) = 0$.

(3) There exists an exact sequence

$$0 \to M \to C \otimes_R P_n \to \cdots \to C \otimes_R P_1 \to M \to 0$$

with each P_i projective such that the sequence is $\mathrm{Hom}_R(-, F)$-exact for every module F with $\mathcal{F}_C$-pd_R $(F) < \infty$.

(4) There exists an exact sequence $0 \to M \to C \otimes_R P_n \to \cdots \to C \otimes_R P_1 \to M \to 0$, where each P_i is projective, and there exists a positive integer k such that

$$\mathrm{Ext}_R^{k+1}(M, C \otimes_R Q) = \mathrm{Ext}_R^{k+2}(M, C \otimes_R Q) = \cdots = \mathrm{Ext}_R^{k+n}(M, C \otimes_R Q) = 0$$

for every $C \otimes_R Q \in \mathcal{F}_C(R)$.

(5) There exists an exact sequence $0 \to M \to C \otimes_R P_n \to \cdots \to C \otimes_R P_1 \to M \to 0$, where each P_i is projective, and there exists a positive integer k such that

$$\mathrm{Ext}_R^{k+1}(M, N) = \mathrm{Ext}_R^{k+2}(M, N) = \cdots = \mathrm{Ext}_R^{k+n}(M, N) = 0$$

for every module N with $\mathcal{F}_C$-$\mathrm{pd}_R(N) < \infty$.

Proof (1) ⇔ (2) and (3) ⇒ (1) are obvious by Definition 3.1.

(1) ⇒ (3): Let M be an n-strongly D_C-projective module. Then there exists an exact sequence

$$\mathcal{P} = 0 \to M \to C \otimes_R P_n \to \cdots \to C \otimes_R P_1 \to M \to 0$$

such that $\mathcal{P}$ is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, where each P_i is projective. Let F be a module and $m \geq 1$ a positive integer such that $\mathcal{F}_C$-$\mathrm{pd}_R(F) = m < \infty$. Then there is an exact sequence $0 \to K \to V \to F \to 0$ with $V \in \mathcal{F}_C(R)$ and $\mathcal{F}_\mathcal{C}$-$\mathrm{pd}_R(K) = m - 1$. Since every n-strongly D_C-projective module is D_C-projective by Proposition 3.6(3), it follows that $\mathrm{Ext}_R^{\geq 1}(M, K) = 0 = \mathrm{Ext}_R^{\geq 1}(M, V)$ by Lemma 3.3 and [29, Proposition 1.5].

Moreover, since $\mathcal{F}_C(R) \subseteq \mathcal{B}_C(R)$ by [19, Corollary 6.1] and $\mathcal{B}_C(R) \subseteq \mathcal{P}_C(R)^{\perp}$ by [15, Corollary 3.2(2)], we have $\mathrm{Ext}_R^j(C \otimes_R P_i, K) = 0$ for $1 \leq i \leq n$ by dimension shifting. This implies that the sequence of complexes

$$0 \to \mathrm{Hom}_R\ (\mathcal{P},\ K) \to \mathrm{Hom}_R\ (\mathcal{P}\ ,V) \to \mathrm{Hom}_R\ (\mathcal{P},\ F) \to 0$$

is exact. Moreover, since the complexes $\mathrm{Hom}_R(\mathcal{P},K)$ and $\mathrm{Hom}_R(\mathcal{P},V)$ are exact by inductive hypothesis, we conclude that the complex $\mathrm{Hom}_R(\mathcal{P},F)$ is exact.

(2) $\Leftrightarrow$ (4). Assume that M is n-strongly D_C-projective, then there exists an exact sequence

$$\mathcal{P} = 0 \to M \to C \otimes_R P_n \to \cdots \to C \otimes_R P_1 \to M \to 0$$

with each P_i projective such that $\mathrm{Ext}_R^{\geq 1}(M, \mathcal{F}_C(R)) = 0$. Since $\mathcal{F}_C(R) \subseteq \mathcal{B}_C(R)$ by [19, Corollary 6.1] and $\mathcal{B}_C(R) \subseteq \mathcal{P}_C(R)^{\perp}$ [15, Corollary 3.2(2)] by dimension shifting, we have the isomorphism

$$\mathrm{Ext}_R^k(M, C \otimes_R Q) \cong \mathrm{Ext}_R^{n+k}(M, C \otimes_R Q)$$

for all $k \geq 1$, where Q is a flat R-module. Therefore, if the n successive terms of $\mathrm{Ext}_R^n(M, C \otimes_R Q)$ is zero, then we have $\mathrm{Ext}_R^k(M, C \otimes_R Q) = 0$ with Q an flat R-module and $k \geq 1$.

(3) $\Leftrightarrow$ (5). The proof is similar to that of (2) $\Leftrightarrow$ (4).

Corollary 3.5[30, Theorem 3.2] Let M be an R-module. Then the following are equivalent

(1) M is strongly Ding projective.

(2) There exists an exact sequence $0 \to M \to P \to M \to 0$ with P projective such that $\mathrm{Ext}_R^{\geq 1}(M, \mathcal{F}(R)) = 0$.

(3) There exists an exact sequence $0 \to M \to P \to M \to 0$ with P projective such that the short sequence $0 \to \mathrm{Hom}_R(M, F) \to \mathrm{Hom}_R(P, F) \to \mathrm{Hom}_R(M, F) \to 0$ is exact for every flat R-module F.

(4) There exists an exact sequence $0 \to M \to P \to M \to 0$ with P projective such that the short sequence $0 \to \mathrm{Hom}_R(M, F) \to \mathrm{Hom}_R(P, F) \to \mathrm{Hom}_R(M, F) \to 0$ is exact for every R-module F with finite flat dimension.

The next proposition reveals the relations between strongly D_C-projective module, n-strongly D_C-projective modules and D_C-projective modules.

Proposition 3.6 Let n and m be two positive integers. Then the following statements hold

(1) Every strongly D_C-projective module is n-strongly D_C-projective.

(2) If $m = kn$, then every n-strongly D_C-projective module is m-strongly D_C-projective.

(3) Every n-strongly D_C-projective module is D_C-projective.

(4) For any $n \geq 1$, the class n-$\mathcal{SDP}_C(R)$ is closed under direct sums.

(5) If M is n-strongly D_C-projective and $K_i = \mathrm{Ker}(C \otimes_R P_i \to C \otimes_R P_{i-1})$ with $1 \leq i \leq n$ (put $M = C \otimes_R P_0$), then $\bigoplus\limits_{i=1}^{n} K_i$ is strongly D_C-projective.

Propostition 3.6 Let R be a coherent ring. Then an R-module M is D_C-projective if and only if it is a direct summand of a strongly D_C-projective module.

Proof The proof is similar to that of [3, Theorem 2.7].

Lemma 3.8 Let m and n be two positive integers. Then we have

(1) If $m = kn$, then $m\text{-}\mathcal{SDP}_C(R) \bigcap n\text{-}\mathcal{SDP}_C(R) = n\text{-}\mathcal{SDP}_C(R)$,

(2) If $m = kn+i$ with k a positive integer and $0 < i < n$, then $m\text{-}\mathcal{SDP}_C(R) \bigcap n\text{-}\mathcal{SDP}_C(R) \subseteq i\text{-}\mathcal{SDP}_C(R)$.

Proof (1) follows directly from Proposition 3.6.

(2) The proof is a modification of that of [28, Proposition 3.4]. Note that $m\text{-}\mathcal{SDP}_C(R) \bigcap n\text{-}\mathcal{SDP}_C(R) \subseteq m\text{-}\mathcal{SDP}_C(R) \bigcap kn\text{-}\mathcal{SDP}_C(R)$ by Proposition 3.6. Let $M \in m\text{-}\mathcal{SDP}_C(R) \bigcap kn\text{-}\mathcal{SDP}_C(R)$. Then there exists an exact sequence $\mathcal{P}= 0 \to M \to C\otimes_R P_m \to \cdots \to C\otimes_R P_1 \to M \to 0$ such that $\mathcal{P}$ is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, where each P_i is projective. Let $H_i = \mathrm{Ker}(C\otimes_R P_i \to C\otimes_R P_{i-1})$ for $2 \le i \le m$. Since $M \in kn\text{-}\mathcal{SDP}_C(R)$ and $\mathcal{P}_C(R)$ is closed under direct sums, there exist $C \otimes_R \widetilde{P}$ and $C \otimes_R \widehat{P}$ with $\widetilde{P}$ and $\widehat{P}$ projective such that

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C\otimes_R \widehat{P} \arrow[r, equal] \arrow[d] & C\otimes_R \widehat{P} \arrow[d] & \\
0 \arrow[r] & H_{kn+1} \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M\oplus(C\otimes_R \widetilde{P}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{kn+1} \arrow[r] & C\otimes_R P_{kn+1} \arrow[r] \arrow[d] & H_{kn} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $C\otimes_R \widetilde{P}, C\otimes_R \widehat{P} \in \mathcal{P}_C(R)$ and $\mathcal{P}_C(R)$ is closed under extensions by [19, Proposition 5.2], we have $X \in \mathcal{P}_C(R)$. Consider the following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H_{kn+1} \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{kn+1} \arrow[r] & X \arrow[r] \arrow[d] & M\bigoplus(C\otimes_R \widetilde{P}) \arrow[r] \arrow[d] & 0 \\
 & & C\otimes_R \widetilde{P} \arrow[r, equal] \arrow[d] & C\otimes_R \widetilde{P} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Because $X \in \mathcal{P}_C(R)$ and $\mathcal{P}_C(R)$ is projectively resolving by [19, Corollary 6.4], we have $Y \in \mathcal{P}_C(R)$. Now combining the sequence $\mathcal{P}$ and the first row in the above diagram, we obtain the following exact sequence:

$$0 \to M \to C\otimes_R P_m \to \cdots \to C\otimes_R P_{kn+2} \to Y \to M \to 0,$$

which is $\mathrm{Hom}_R(-,\mathcal{F}_C(R))$-exact. This implies that M is i-strongly D_C-projective. Therefore, m-$\mathcal{SDP}_C(R)\bigcap n$-$\mathcal{SDP}_C(R)\subseteq i$-$\mathcal{SDP}_C(R)$.

Proposition 3.9 m-$\mathcal{SDP}_C(R)\bigcap n$-$\mathcal{SDP}_C(R)=(m,n)$-$\mathcal{SDP}_C(R)$, where (m,n) is the greatest common divisor of m and n.

Proof The implication (m,n)-$\mathcal{SDP}_C(R)\subseteq m$-$\mathcal{SDP}_C(R)\bigcap n$-$\mathcal{SDP}_C(R)$ is obvious by Proposition 3.6(2). It suffices to prove the converse inclusion. If $m=nq_0+r_0$ with $0<r_0<n$, then m-$\mathcal{SDP}_C(R)\bigcap n$-$\mathcal{SDP}_C(R)\subseteq r_0$-$\mathcal{SDP}_C(R)$ by Lemma 3.8. If $n=r_0q_1+r_1$ with $0<r_1<r_0$, then r_0-$\mathcal{SDP}_C(R)\bigcap n$-$\mathcal{SDP}_C(R)\subseteq r_1$-$\mathcal{SDP}_C(R)$ again by Lemma 3.8. Therefore, m-$\mathcal{SDP}_C(R)\bigcap n$-$\mathcal{SDP}_C(R)\subseteq r_1$-$\mathcal{SDP}_C(R)$. Continuing this process after finite steps, there is a positive integer s with $r_{s-1}=(m,n)q_{s+1}$ such that m-$\mathcal{SDP}_C(R)\bigcap n$-$\mathcal{SDP}_C(R)=(m,n)$-$\mathcal{SDP}_C(R)$.

Theorem 3.10 Let M be an R-module and $n\geq 1$. Then the following statements are equivalent

(1) M is an n-strongly D_C-projective module.

(2) There exists an exact sequence $0\longrightarrow M\xrightarrow{\alpha_{n+1}}C\otimes_R P_n\xrightarrow{\alpha_n}\cdots\xrightarrow{\alpha_2}C\otimes_R P_1\xrightarrow{\alpha_1}M\longrightarrow 0$ of R-modules with each P_i projective for any $1\leq i\leq n$, such that $\bigoplus\limits_{i=1}^{n}\mathrm{Ker}\alpha_{\mathrm{i}}$ is strongly D_C-projective.

(3) There exists an exact sequence $0\longrightarrow M\xrightarrow{\alpha_{n+1}}C\otimes_R P_n\xrightarrow{\alpha_n}\cdots\xrightarrow{\alpha_2}C\otimes_R P_1\xrightarrow{\alpha_1}M\longrightarrow 0$ of R-modules with each P_i projective for any $1\leq i\leq n$, such that $\bigoplus\limits_{i=1}^{n}\mathrm{Ker}\alpha_{\mathrm{i}}$ is D_C-projective.

(4) There exists an exact sequence $0\longrightarrow M\xrightarrow{\alpha_{n+1}}H_n\xrightarrow{\alpha_n}\cdots\xrightarrow{\alpha_2}H_1\xrightarrow{\alpha_1}M\longrightarrow 0$ of R-modules with $\mathcal{P}_{\mathcal{C}}$-$\mathrm{pd}_R(H_i)<\infty$ for any $1\leq i\leq n$, such that $\bigoplus\limits_{i=1}^{n}\mathrm{Ker}\alpha_{\mathrm{i}}$ is strongly D_C-projective.

(5) There exists an exact sequence $0\longrightarrow M\xrightarrow{\alpha_{n+1}}H_n\xrightarrow{\alpha_n}\cdots\xrightarrow{\alpha_2}H_1\xrightarrow{\alpha_1}M\longrightarrow 0$ of R-modules with $\mathcal{P}_{\mathcal{C}}$-$\mathrm{pd}_R(H_i)<\infty$ for any $1\leq i\leq n$, such that $\bigoplus\limits_{i=1}^{n}\mathrm{Ker}\alpha_{\mathrm{i}}$ is D_C-projective.

Proof (1) $\Rightarrow$ (2) follows from Proposition 3.6(5). (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial by Proposition 3.6. It suffices to prove (5) $\Rightarrow$ (1). Suppose that there exists an exact sequence

$$0\longrightarrow M\xrightarrow{\alpha_{n+1}}H_n\xrightarrow{\alpha_n}\cdots\xrightarrow{\alpha_2}H_1\xrightarrow{\alpha_1}M\longrightarrow 0$$

of R-modules with $\mathcal{P}_C$-$\mathrm{pd}_R(H_i)<\infty$ for any $1\leq i\leq n$, such that $\bigoplus\limits_{i=1}^{n}\mathrm{Ker}\alpha_{\mathrm{i}}$ is D_C-projective. For each $1\leq i\leq n$, we have the exact sequence $0\to\mathrm{Ker}\alpha_{\mathrm{i}}\to H_i\to\mathrm{Ker}\alpha_{\mathrm{i-1}}\to 0$. Since $\bigoplus\limits_{i=1}^{n}\mathrm{Ker}\alpha_{\mathrm{i}}$ is D_C-projective, it follows that each $\mathrm{Ker}\alpha_{\mathrm{i}}$ is D_C-projective by Lemma 3.3 and [29, Theorem 1.12]. Therefore, each H_i is D_C-projective for all i by Lemma 3.3 and [29, Proposition 1.10]. For each i, there exists an exact sequence of R-modules

$$0\to N_i\to C\otimes_R P\to H_i\to 0$$

such that P is projective and $\mathcal{P}_C-\mathrm{pd}_R(N_i)<\infty$. This implies that the above sequence is split by [29, Proposition 1.5], and thus each H_i is C-projective. In particular, M is D_C-projective by [29, Proposition 1.8 and Theorem 1.12]. Therefore, we have $\mathrm{Ext}_R^{\geq 1}(M,\mathcal{F}_C(R))=0$ by [29, Proposition 1.4]. This implies that M is n-strongly D_C-projective by Theorem 3.4.

§4. Strongly D_C-flat Modules

Recall that an R-module M is n-strongly Gorenstein flat [28], if there exists an exact sequence $0\to M\to F_n\to\cdots\to F_1\to M\to 0$ with each F_i flat such that $I\otimes_R -$ leaves the sequence exact whenever I is an injective R-module.

More generally, we give the following

Definition 4.1 An R-module M is said to be an n-strongly D_C-flat module, if there exists an exact sequence

$$0\to M\to C\otimes_R F_n\to\cdots\to C\otimes_R F_1\to M\to 0$$

with each F_i flat for any $1\leq i\leq n$, such that the sequence is $\mathcal{FI}_C(R)\otimes_R -$exact.

We denote by n-$\mathcal{SDF}_C(R)$ the subcategory of n-strongly D_C-flat R-modules relative to the semidualizing module C. Recall that an R-module M is called D_C-flat, if there exists an exact sequence of C-flat R-modules

$$\cdots\to C\otimes_R F_1\to C\otimes_R F_0\to C\otimes_R F^0\to C\otimes_R F^1\to\cdots$$

with $M\cong\mathrm{Coker}(C\otimes_R F_1\to C\otimes_R F_0)$ such that the sequence is $\mathcal{FI}_C(R)\otimes_R -$ exact, where all F_i, F^i are flat R-modules.

Remark 4.2 (1) If $C=R$, it is easy to see that D_C-flat modules are just the class of Ding flat modules, and n-strongly D_C-flat modules are precisely the class of n-strong Ding flat modules. Clearly, every n-strongly D_C-flat module is D_C-flat and every n-strong Ding flat module is Ding flat.

(2) We call a 1-strongly D_C-flat module a strongly D_C-flat module, for simplicity. In particular, a 1-strongly D_R-flat module is called to be a strongly Ding flat module. Clearly, every strongly D_C-flat module is n-strongly D_C-flat.

Definition 4.3 An R-module M is said to be n-strongly D_C-injective, if there exists an exact sequence

$$0\to M\to\mathrm{Hom}_R(C,I_1)\to\cdots\to\mathrm{Hom}_R(C,I_n)\to M\to 0$$

with each I_i injective for any $1\leq i\leq n$ such that the sequence is $\mathrm{Hom}_R(\mathcal{FI}_C(R),-)-$ exact.

It was shown in [28, Proposition 4.9] that if M is an n-strongly Gorenstein flat module, the M^+ is n-strongly Gorenstein injective. Similarly, we have the following

Proposition 4.4 If M is an n-strongly D_C-flat module, then M^+ is n-strongly D_C-injective.

Proof Suppose that M is n-strongly D_C-flat. Then there exists an exact sequence

$$\mathcal{F} = 0 \to M \to C \otimes_R F_n \to \cdots \to C \otimes_R F_1 \to M \to 0$$

with each F_i flat such that $\mathcal{F}$ is $\mathcal{FI}_C(R)\otimes_R-$ exact, and thus $\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C,I),M)=0$ for every FP-injective R-module I. Therefore, we have the following exact sequence

$$\mathcal{F}^+ = 0 \to M^+ \to \mathrm{Hom}_R(C,F_1^+) \to \cdots \to \mathrm{Hom}_R(C,F_n^+) \to M^+ \to 0$$

with each F_i^+ injective. Moreover, we have the isomorphism

$$\mathrm{Ext}^i_R(\mathrm{Hom}_R(C,E),M^+) \cong \mathrm{Tor}^R_i(\mathrm{Hom}_R(C,E),M)^+ = 0$$

for any $i \geq 1$ by [4, Chapter VI, Proposition 5.1], where $\mathrm{Hom}_R(C,E) \in \mathcal{FI}_C(R)$. This implies that the exact sequence $\mathcal{F}^+$ is $\mathrm{Hom_R}(\mathcal{FI}_C(R),-)-$ exact, and thus M^+ is n-strongly D_C-injective.

Lemma 4.4 Let R be a coherent ring. Then an R-module M is C-FP-injective if and only if its character module M^+ is C-flat.

Proof Assume that M is C-FP-injective. Then we can write M as $\mathrm{Hom}_R(C,E)$ for some FP-injective R-module E. Since C has a degreewise finite projective resolution and $\mathbb{Q}/\mathbb{Z}$ is injective, we have the following

$$M^+ = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(C,E),\mathbb{Q}/\mathbb{Z}\big) \cong \mathbb{C}\otimes_R\mathrm{Hom}_{\mathbb{Z}}\big(E,\mathbb{Q}/\mathbb{Z}\big)$$

by the Hom-evaluation isomorphism. Moreover, E^+ is a flat module by [5, Theorem 1] since R is coherent, and hence M^+ is C-flat. Conversely, if M^+ is C-flat then $M^+ \in \mathcal{B}_C(R)$ and $\mathrm{Hom}_R(C,M^+) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}\otimes_R C,\mathbb{Q}/\mathbb{Z})$ is a flat R-module by [19, Lemma 5.1(a)]. This implies that $M\otimes_R C$ is a FP-injective module by [5, Theorem 1]. Since $M^+ \in \mathcal{B}_C(R)$, we have the isomorphisms

$$\begin{aligned} M^+ &\cong C\otimes_R\mathrm{Hom}_R(C,M^+) \cong C\otimes_R\mathrm{Hom}_R(M\otimes_R C,\mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(C,M\otimes_R C),\mathbb{Q}/\mathbb{Z}\big). \end{aligned}$$

Here the second isomorphism follows from the adjoint isomorphism and the third one is the Hom-evaluation isomorphism since C has a degreewise finite projective resolution and $\mathbb{Q}/\mathbb{Z}$ is injective. This implies that $M \cong \mathrm{Hom}_R(C,M\otimes_R C)$ is C-FP-injective.

The next proposition shows that n-strongly D_C-projective modules over coherent rings are indeed stronger than n-strongly D_C-flat modules.

Proposition 4.6 Let R be a coherent ring. Then every n-strongly D_C-projective R-module is n-strongly D_C-flat.

Proof Suppose that M is an n-strongly D_C-projective R-module. Then there exists an exact sequence of R-modules

$$\mathcal{P} = 0 \to M \to C\otimes_R P_n \to \cdots \to C\otimes_R P_1 \to M \to 0$$

such that $\mathcal{P}$ is $\mathrm{Hom_R}(-,\mathcal{F}_C(R))$-exact, where each P_i is projective. For any C-FP-injective R-module $\mathrm{Hom}_R(C,I)$, we note that $\mathrm{Hom}_R(C,I)^+$ is C-flat by Lemma 4.5. Now applying the functor $\mathrm{Hom}_R\big(-,\mathrm{Hom}_R(C,I)^+\big)$ to the exact sequence $\mathcal{P}$, we get the exact sequence

$$0 \to \mathrm{Hom}_R\big(M, \mathrm{Hom}_R(C,I)^+\big) \to \mathrm{Hom}_R\big(C \otimes_R P_1, \mathrm{Hom}_R(C,I)^+\big) \to \cdots \to \mathrm{Hom}_R\big(C \otimes_R P_n, \mathrm{Hom}_R(C,I)^+\big) \to \mathrm{Hom}_R\big(M, \mathrm{Hom}_R(C,I)^+\big) \to 0,$$

which gives the exactness of the sequence

$$0 \to \big(\mathrm{Hom}_R(C,I) \otimes_R M\big)^+ \to \big(\mathrm{Hom}_R(C,I) \otimes_R (C \otimes_R P_1)\big)^+ \to \cdots \to \big(\mathrm{Hom}_R(C,I) \otimes_R (C \otimes_R P_n)\big)^+ \to \big(\mathrm{Hom}_R(C,I) \otimes_R M\big)^+ \to 0.$$

This implies that the following sequence

$$0 \to \mathrm{Hom}_R(C,I) \otimes_R M \to \mathrm{Hom}_R(C,I) \otimes_R (C \otimes_R P_n) \to \cdots \to \mathrm{Hom}_R(C,I) \otimes_R (C \otimes_R P_1) \to \mathrm{Hom}_R(C,I) \otimes_R M \to 0$$

is exact. Therefore, M is an n-strongly D_C-flat module.

Corollary 4.7 Let R be a coherent ring. Then every n-strongly Ding projective module is n-strongly Gorenstein flat.

Let R and S be two commutative rings. Recall that a ring homomorphism $\varphi : R \to S$ is a flat ring homomorphism if S is a flat R-module.

Lemma 4.8[25, Theorem 3.4.1] Let C be a semidualizing R-module and let $\varphi : R \to S$ be a ring homomorphism. Then $S \in \mathcal{A}_C(R)$ if and only if $S \otimes_R C$ is a semidualizing S-module with $\mathrm{Tor}_i^R(C,S) = 0$ for all $i \geq 1$.

Lemma 4.9 Let $\varphi : R \to S$ be a flat ring homomorphism. If M is FP-injective as an S-module, then M is FP-injective as an R-module.

Proof Let N be a finitely presented R-module. Then there is an exact sequence $0 \to K \to P \to N \to 0$ of R-modules with K finitely generated and P finitely presented projective. Since $\varphi : R \to S$ is a flat ring homomorphism, we have the following exact sequence of S-modules

$$0 \to K \otimes_R S \to P \otimes_R S \to N \otimes_R S \to 0.$$

It is clear that $K \otimes_R S$ is finitely generated and $P \otimes_R S$ is finitely generated projective. This implies that $N \otimes_R S$ is finitely presented, and thus $\mathrm{Ext}_S^1(N \otimes_R S, M) = 0$ since M is FP-injective as an S-module. Therefore, we have $\mathrm{Ext}_R^1(N, M) = 0$ by [20, Theorem 11.65]. It follows that M is an FP-injective R-module, as desired.

Next we investigate the property of n-strongly D_C-flat modules under change of rings.

Proposition 4.10 Let $\varphi : R \to S$ be a flat ring homomorphism. If $M \in n\text{-}\mathcal{SDF}_C(R)$, then $S \otimes_R M \in n\text{-}\mathcal{SDF}_{S \otimes_R C}(S)$.

Proof Assume that $M \in n\text{-}\mathcal{SDF}_C(R)$. Then there exists an exact sequence

$$\mathcal{F} = 0 \to M \to C \otimes_R F_n \to \cdots \to C \otimes_R F_1 \to M \to 0$$

with each F_i flat as R-module such that $\mathcal{F}$ is $\mathcal{FI}_C(R) \otimes_R -$ exact. Therefore, we have $\mathrm{Tor}_i^R(\mathrm{Hom}_R(C,I), M) = 0$ for every FP-injective R-module I. Since S is a flat R-module, the complex

$$S \otimes_R \mathcal{F}= 0 \to S \otimes_R M \to S\otimes_R(C \otimes_R F_n) \to \cdots \to S\otimes_R(C \otimes_R F_1) \to S \otimes_R M \to 0$$

is exact, and thus the complex

$$S\otimes_R \mathcal{F}= 0 \to S\otimes_R M \to (S\otimes_R C)\otimes_S(S\otimes_R F_n) \to \cdots \to(S\otimes_R C)\otimes_S (S\otimes_R F_1) \to S\otimes_R M \to 0$$

is exact. It is easy to see that every $S \otimes_R F_i$ is flat as S-module for each i. Since $S \in \mathcal{A}_C(R)$, $S \otimes_R C$ is a semidualizing S-module by Lemma 4.8. It suffices to show that the complex $\mathrm{Hom}(S \otimes_R C, E)\otimes_S(S \otimes_R \mathcal{F})$ is exact for every FP-injective S-module E. In fact, we have

$$\mathrm{Hom_S}\big(S \otimes_R C, E\big) \otimes_S \big(S \otimes_R \mathcal{F}\big)\cong \mathrm{Hom_R}\big(C, \mathrm{Hom_S}(S, E)\big)\otimes_R\mathcal{F}$$

since $E \cong\mathrm{Hom}_S(S, E)$ and E is an FP-injective S-module, it follows that $\mathrm{Hom}_S(S, E)$ is an FP-injective R-module by Lemma 4.9. Now the result follows from the above isomorphism, and hence $S \otimes_R M \in n\text{-}\mathcal{SDF}_{S\otimes_R C}(S)$.

Corollary 4.11 If $\varphi : R \to S$ is a flat ring homomorphism, then $n\text{-}\mathcal{SDF}_{S\otimes_R C}\text{-pd}_S(S \otimes_R M) \leq n\text{-}\mathcal{SDF}_C\text{-pd}_R(M)$.

Proposition 4.12 Let S be a multiplicatively closed set of a ring R. If M is n-strongly D_C-flat, then $S^{-1}M$ is n-strongly D_C-flat.

Proof Assume that M is n-strongly D_C-flat. Then there exists an exact sequence

$$\mathcal{F}= 0 \to M \to C\otimes_R F_n \to \cdots \to C\otimes_R F_1 \to M \to 0$$

such that $\mathcal{F}$ is $\mathcal{FI}_C(R) \otimes_R -$ exact, where each F_i is flat. Since $S^{-1}R$ is flat as an R-module, the following sequence

$$S^{-1}R\otimes_R\mathcal{F}= 0 \to S^{-1}R \otimes_R M\to S^{-1}R\otimes_R(C \otimes_R F_n) \to \cdots \to S^{-1}R\otimes_R(C \otimes_R F_1) \to S^{-1}R \otimes_R M\to 0$$

is exact. For each i, we have the isomorphisms $S^{-1}R \otimes_R (C \otimes_R F_i) \cong C \otimes_R S^{-1}R \otimes_R F_i \cong C \otimes_R S^{-1}F_i$. Therefore, we have the exact sequence

$$S^{-1}\mathcal{F}= 0 \to S^{-1}M\to C \otimes_R S^{-1}F_n\to \cdots \to C \otimes_R S^{-1}F_1 \to S^{-1}M \to 0.$$

Now it suffices to prove that the sequence $S^{-1}\mathcal{F}$ is $\mathcal{FI}_C(R) \otimes_R -$ exact. In fact, for every $\mathrm{Hom}_R(C, E) \in \mathcal{FI}_C(R)$, we have the isomorphisms

$$\mathrm{Hom}_R(C, E) \otimes_R S^{-1}\mathcal{F} \cong \mathrm{Hom}_R(C, E) \otimes_R S^{-1}R\otimes_R\mathcal{F} \cong \mathrm{Hom}_R(C, E)\otimes_R\mathcal{F}\otimes_R S^{-1}R,$$

which implies that $S^{-1}\mathcal{F}$ is $\mathcal{FI}_C(R) \otimes_R -$ exact. Therefore, $S^{-1}M$ is n-strongly D_C-flat.

Corollary 4.13 Let S be a multiplicatively closed set of a ring R. Then

(1) If M is n-strongly Ding flat, then $S^{-1}M$ is n-strongly Ding flat,

(2) If M is strongly Ding flat, then $S^{-1}M$ is strongly Ding flat,

(3) If M is strongly D_C-flat, then $S^{-1}M$ is D_C-flat.

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