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Batalin-Vilkovisky Structure on Hochschild Cohomology of Self-Injective Quadratic Monomial Algebras

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Abstract: We give a complete description of the Batalin-Vilkovisky algebra structure on Hochschild cohomology of the self-injective quadratic monomial algebras.

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§1. Introduction

The Hochschild homology and cohomology theory played a fundamental role in representation theory of artin algebras. Hochschild homology is closely related to the voriented cycle and the global dimension of algebras; Hochschild cohomology is closely related to simple connectedness, separability and deformation theory.

The monomial algebras is a class of relatively simple algebras. The Hochschild homology and cohomology of this kind of algebras have been widely studied. The cup product on the Hochschild cohomology has been described for some especial monomial algebras such as radical square zero algebras [3], exterior algebras [14], truncated quiver algebras [1, 7] and so on. For the Lie structure on the Hochschild cohomology, Xu and Zhang have described the Gerstenhaber bracket on the Hochschild cohomology of truncated quiver algebras in terms of parallel paths [15]. The Gerstenhaber bracket on the Hochschild cohomology of triangular quadratic monomial algebras are considered in [2]. However, for most finite dimensional algebras, it is little to known about the the Gerstenhaber bracket of the Hochschild cohomology. Here we will give the description of the Gerstenhaber bracket on Hochschild cohomology of the self-injective quadratic monomial algebras clearly.

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During several decades, a new structure in Hochschild theory has been extensively studied in topology and mathematical physics, and recently this was introduced into algebra, the so-called Batalin-Vilkovisky structure. Roughly speaking a Batalin-Vilkovisky structure is an operator on a Gerstenhaber algebra which squares to zero, and which together with the cup product, can express the Gerstenhaber bracket. A Batalin-Vilkovisky algebra structure exists only on Hochschild cohomology of certain special classes of algebras. Tradler found that the Hochschild cohomology of a finite-dimensional self-injective algebra is a Batalin-Vilkovisky algebra [11]. In 2016, Lambre et al. and Volkov independently showed that this result is also valid for Frobenius algebras with semisimple Nakayama automorphisms [8, 13]. But it is very difficult to give the Batalin-Vilkovisky algebra structure on the Hochschild cohomology of an algebra in general. Here for the self-injective quadratic monomial algebras, we can give a complete description of the Batalin-Vilkovisky algebra structure on the Hochschild cohomology of this kind of algebras.

The paper is structured as follows. In Section 2, we review the definitions of Hochschild cohomology, cup product, Gerstenhaber bracket product and Batalin-Vilkovisky algebra. In Section 3, we first show that the self-injective quadratic monomial algebras are essentially the radical square zero Nakayama algebras of type $\widetilde{\mathbf{A}}$. We denote this class of algebras by A_n and recall the minimal projective bimodule resolution of A_n which has given by Bardezll. We also give a basis of each degree of Hochschild cohomology of A_n by using this resolution. In Section 4, we give the ring structure on $HH^*(A_n)=\bigoplus_{m\geq 0}HH^m(A_n)$. In particular, we show that the Hochschild cohomology ring modulo the nilpotent ideal is finite generated, and so give a positive answer to the Snashall-Solberg conjecture. In Section 5, by the chain mappings between the reduced bar resolution and the minimal projective bimodule resolution of A_n, we give the Gerstenhaber algebra structure and the Batalin-Vilkovisky algebra structure on $HH^*(A_n)$ clearly. Throughout this paper, we fix $\Bbbk$ an algebraically closed field with char$\Bbbk=0$, $\otimes:=\otimes_{\Bbbk}$.

§2. Hochschild cohomology of associative algebras

The cohomology theory of associative algebras was introduced by Hochschild (see [6]). Let Λ be an associative algebra over a field $\Bbbk$. The Hochschild cohomology $HH^*(\Lambda)$ of Λ has a very rich structure. In this section, we recall the cup product, the Gerstenhaber bracket and Batalin-Vilkovisky algebra structure in the Hochschild cohomology.

For an associative $\Bbbk$-algebra Λ, there is a projective bimodule resolution of Λ as following:

$$\mathbb{B}=(B_m,d_m): \qquad \cdots\longrightarrow \Lambda^{\otimes(m+2)}\xrightarrow{d_m}\Lambda^{\otimes(m+1)}\longrightarrow\cdots\longrightarrow\Lambda^{\otimes 3}\xrightarrow{d_1}\Lambda^{\otimes 2}\xrightarrow{d_0}\Lambda\longrightarrow 0,$$

where d_0 is the multiplication map, $B_m=A^{\otimes(m+2)}$ for $m\geq 0$, and d_m is defined by

$$d_m(a_0\otimes a_1\otimes\cdots\otimes a_{m+1})=\sum_{i=0}^{m}(-1)^i a_0\otimes\cdots\otimes a_{i-1}\otimes a_ia_{i+1}\otimes a_{i+2}\otimes\cdots\otimes a_{m+1},$$

for all $a_0,a_1,\cdots,a_{m+1}\in\Lambda$. This resolution is called the bar resolution of Λ.

Let $\{e_1,e_2,\cdots,e_l\}$ be a complete set of primitive orthogonal idempotents of Λ, E the subalgebra of Λ generated by $\{e_1,e_2,\cdots,e_l\}$. Denote $\bar{\Lambda}=\Lambda/E$, the quotient $\Bbbk$-module, and $\bar{B}_m=\Lambda\otimes_E\bar{\Lambda}^{\otimes_E m}\otimes_E\Lambda$. Then the quotients $\bar{B}_m$ constitute a complex $\bar{\mathbb{B}}=(\bar{B}_m,\bar{d}_m)$, where the differential $\bar{d}_m$ induced from d_m, for all $m\geq 0$. The complex $\bar{\mathbb{B}}$ is also a projective bimodule resolution of Λ, and is called the reduced bar resolution of Λ.

Applying the functor $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to the complex $\mathbb{B}$ (or $\bar{\mathbb{B}}$), we get a complex $\mathrm{Hom}_{\Lambda^e}(\mathbb{B},\Lambda)$. Note that for each $m\geq 0$, $\mathrm{Hom}_{\Lambda^e}(B_m,\Lambda)\cong\mathrm{Hom}_{\Bbbk}(\Lambda^{\otimes m},\Lambda)$, the Hochschild cohomology of Λ is just the homology of complex $\mathbb{C}=(C^m,\delta^m)$, where $C^m=\mathrm{Hom}_{\Bbbk}(\Lambda^{\otimes m},\Lambda)$ and

$$\begin{aligned}\delta^m(f)(a_1\otimes\cdots\otimes a_{m+1})=&a_1f(a_2\otimes\cdots\otimes a_{m+1})\\&+\sum_{i=1}^{m}(-1)^if(a_1\otimes\cdots\otimes a_{i-1}\otimes a_ia_{i+1}\otimes a_{i+2}\otimes\cdots\otimes a_{m+1})\\&+(-1)^{m+1}f(a_1\otimes\cdots\otimes a_m)a_{m+1},\end{aligned}$$

for any $f\in\mathrm{Hom}_{\Bbbk}(\Lambda^{\otimes m},\Lambda)$, and $a_1\otimes\cdots\otimes a_{m+1}\in\Lambda^{\otimes(m+1)}$.

The cup product $\alpha\sqcup\beta\in C^{m+l}(\Lambda)=\mathrm{Hom}_{\Bbbk}(\Lambda^{\otimes(m+l)},\Lambda)$ for $\alpha\in C^m(\Lambda)$ and $\beta\in C^l(\Lambda)$ is given by

$$(\alpha\sqcup\beta)(a_1\otimes\cdots\otimes a_{m+l})=\alpha(a_1\otimes\cdots\otimes a_m)\beta(a_{m+1}\otimes\cdots\otimes a_{m+l}).$$

This cup product induces a well-defined product in Hochschild cohomology

$$\sqcup:\quad HH^m(\Lambda)\times HH^l(\Lambda)\longrightarrow HH^{m+l}(\Lambda),$$

which turns the graded $\Bbbk$-vector space $HH^*(\Lambda)=\bigoplus_{i\geq 0}HH^i(\Lambda)$ into a graded commutative algebra, for the details see [6].

Besides addition and multiplication, there is another binary operation on $HH^*(\Lambda)$, which is called Gerstenhaber bracket. Let $\alpha\in C^m(\Lambda)$ and $\beta\in C^l(\Lambda)$. If $m,l\geq 1$, then for $1\leq i\leq m$, define $\alpha\widehat{\circ}_i\beta\in C^{n+m-1}(\Lambda)$ by

$$(\alpha\widehat{\circ}_i\beta)(a_1\otimes\cdots\otimes a_{m+l-1})=\alpha(a_1\otimes\cdots\otimes a_{i-1}\otimes\beta(a_i\otimes\cdots\otimes a_{i+l-1})\otimes a_{i+l}\otimes\cdots\otimes a_{m+l-1}),$$

if $m\geq 1$ and $l=0$, then $\beta\in\Lambda$ and for $1\leq i\leq m$, define

$$(\alpha\widehat{\circ}_i\beta)(a_1\otimes\cdots\otimes a_{m-1})=\alpha(a_1\otimes\cdots\otimes a_{i-1}\otimes\beta\otimes a_i\otimes\cdots\otimes a_{m-1}),$$

for any other case, $\alpha\widehat{\circ}_i\beta=0$. Now we can define the Gerstenhaber bracket. Let

$$\alpha\widehat{\circ}\beta=\sum_{i=1}^{m}(-1)^{(l-1)(i-1)}\alpha\widehat{\circ}_i\beta,$$

and $[\alpha,\beta]=\alpha\widehat{\circ}\beta-(-1)^{(m-1)(l-1)}\beta\widehat{\circ}\alpha$. The above $[\ ,\ ]$ induces a well-defined graded Lie bracket in Hochschild cohomology

$$[\ ,\ ]:\quad HH^m(\Lambda)\times HH^l(\Lambda)\longrightarrow HH^{m+l-1}(\Lambda).$$

This graded Lie bracket is usually called the Gerstenhaber bracket in $HH^{*+1}(\Lambda)$. It is well-known that $(HH^*(\Lambda), \sqcup, [\ ,\ ])$ is a Gerstenhaber algebra (see [5]). That is, the following conditions hold:

(1) $(HH^*(\Lambda), \sqcup)$ is an associative algebra.

(2) $(HH^{*+1}(\Lambda), [\ ,\ ])$ is a graded Lie algebra with bracket $[\ ,\ ]$ of degree -1.

(3) $[f \sqcup g, h] = [f, h] \sqcup g + (-1)^{|f|(|h|-1)} f \sqcup [g, h]$, where $|f|$ denotes the degree of f.

If there is an operator on Hochschild cohomology which squares to zero and together with the cup product can express the Lie bracket, then it is an Batalin-Vilkovisky algebra. Let us review the definition of Batalin-Vilkovisky algebra (see, for example [12]).

Definition 2.1. *A Batalin-Vilkovisky algebra is a Gerstenhaber algebra* $(\Lambda^\bullet, \sqcup, [\ ,\ ])$ *together with an operator* $\Delta : \Lambda^\bullet \to \Lambda^{\bullet-1}$ *of degree* -1 *such that* $\Delta \circ \Delta = 0$ *and*

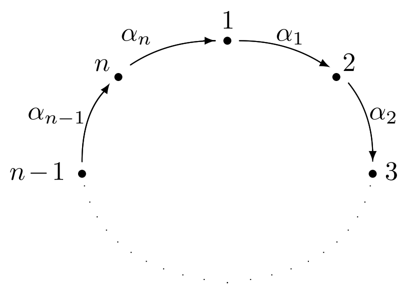
$$[a, b] = -(-1)^{(|a|-1)|b|} \Big(\Delta(a \sqcup b) - \Delta(a) \sqcup b - (-1)^{|a|} a \sqcup \Delta(b) \Big),$$

for homogeneous elements $a, b \in \Lambda^\bullet$.

For any associative $\Bbbk$-algebra with unity, the author proved that $(HH^*(\Lambda), \sqcup, [\ ,\ ])$ is always a Gerstenhaber algebra in [5]. However, for a given algebra, it is very difficult to obtain this structure concretely, that is, to describe exactly the cup product and Gerstenhaber bracket product, is very difficult. The Batalin-Vilkovisky operator Δ does not always exist for the Hochschild cohomology ring $HH^*(\Lambda)$ of an algebra Λ. So far, we only know that there is a Batalin-Vilkovisky operator on Hochschild cohomology ring for few kinds of algebras.

§3. Hochschild cohomology groups

In this section, we consider the Hochschild cohomology groups of the self-injective quadratic monomial algebras. Recently, Lu and Zhu have given a detailed description of self-injective quadratic monomial algebras in [9]. They have shown that a basic $\Bbbk$-algebra Λ over an algebraically closed field $\Bbbk$ is self-injective if and only if Λ is self-injective Nakayama $\Bbbk$-algebra (see Remark 4.3.7 in [9]). Thus by studying the self-injective quadratic monomial algebras, we only need to consider the algebras A_n, which are given by quiver Q as following:



with relations $\alpha_i \alpha_{i+1} = 0$, $i = 1, 2, \cdots, n$, where $\alpha_{n+1} = \alpha_1$. This is, $A_n = \Bbbk Q/I$, where I is an ideal of path algebra $\Bbbk Q$ generated by $\alpha_i \alpha_{i+1} = 0$, $i = 1, 2, \cdots, n$. We denote by e_i the trivial path in Q and the idempotent element in $\Bbbk Q$ corresponding to vertex i, $i=1,2,\cdots,n$. Then $\mathcal{B}:=\{e_i,\alpha_i\,|\,1\le i\le n\}$ is a $\Bbbk$-basis of A_n, and so that $\dim_{\Bbbk}A_n=2n$.

The algebras A_n are truncated quiver algebras. Bardezll and his collaborators have given the Hochschild cohomology groups for truncated quiver algebras in [4]. In this section we will use the parallel paths to give these conclusions again. Firstly, we consider the minimal projective bimodule resolution of A_n. Setting

$$F^0:=\{f_i^0=e_i\},\qquad F^m:=\{f_i^m=\alpha_i\alpha_{i+1}\cdots\alpha_{i+m-1}\,|\,1\le i\le n\},$$

for all $m\ge 1$. We denote by $\mathfrak{o}(p)$ and $\mathfrak{t}(p)$ the originals and terminus of p, for any path $p\in\Bbbk Q$. Let

$$P_m:=\bigoplus_{f\in F^m}A_n\mathfrak{o}(f)\otimes\mathfrak{t}(f)A_n,$$

for $m\ge 0$, and $d_m:P_m\to P_{m-1}$

$$d_m(\mathfrak{o}(f_i^m)\otimes\mathfrak{t}(f_i^m))=\alpha_i\otimes\mathfrak{t}(f_{i+1}^{m-1})+(-1)^m\mathfrak{o}(f_i^{m-1})\otimes\alpha_{i+m-1},$$

for $m\ge 1$. Then we get a minimal projective bimodule resolution of A_n:

$$\mathbb{P}:\ \cdots\longrightarrow P_{m+1}\xrightarrow{d_{m+1}}P_m\longrightarrow\cdots\longrightarrow P_2\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\xrightarrow{d_0}A_n\longrightarrow 0,$$

where d_0 is the multiplication map. For the details see [7]. Applying the functor $\mathrm{Hom}_{A_n^e}(-,A_n)$, we get a complex $\mathrm{Hom}_{A_n^e}(\mathbb{P},A_n)$. The Hochschild cohomology group of A_n is just the homology group of the complex $\mathrm{Hom}_{A_n^e}(\mathbb{P},A_n)$. Now we give an equivalent characterization of the complex $\mathrm{Hom}_{A_n^e}(\mathbb{P},A_n)$.

Let X and Y be the sets of paths in $\Bbbk Q$. We define

$$X//Y:=\{(p,q)\in X\times Y\,|\,\mathfrak{o}(p)=\mathfrak{o}(q)\text{ and }\mathfrak{t}(p)=\mathfrak{t}(q)\},$$

and denote by $\Bbbk\{X//Y\}$ the vector space spanned by the elements in $X//Y$, and call $(p,q)\in\Bbbk Q//\Bbbk Q$ a parallel path. Consider the sets $\mathcal{B}//F^m$, we get

$$\mathcal{B}//F^m=\begin{cases}\{(\alpha_i,f_i^m)\,|\,1\le i\le n\}, & \text{if } m=kn+1,k\in\mathbb{Z};\\ \{(e_i,f_i^m)\,|\,1\le i\le n\}, & \text{if } m=kn,k\in\mathbb{Z};\\ 0, & \text{otherwise}.\end{cases}$$

Define complex $\mathbb{L}=(L^m,\sigma^m)$, where $L^m=\Bbbk(\mathcal{B}//F^m)$ if $m\ge 0$ and for any $m\ge 1$, $\sigma^m:L^{m-1}\to L^m$ is given by

$$\sigma^m(b,f_i^{m-1})=(\alpha_{i-1}b,\,f_{i-1}^m)+(-1)^m(b\alpha_{i+m-1},\,f_i^m).$$

Then we have the following lemma.

Lemma 3.1. $\mathrm{Hom}_{A_n^e}(\mathbb{P},A_n)\cong\mathbb{L}$ *as complexes.*

Proof. We have isomorphisms

$$\begin{aligned}\mathrm{Hom}_{A_n^e}(P_m,A_n)&\cong\bigoplus_{f\in F^m}\mathrm{Hom}_{A_n^e}(A_n\mathfrak{o}(f)\otimes\mathfrak{t}(f)A_n,A_n)\\&\cong\bigoplus_{f\in F^m}\mathfrak{o}(f)A_n\mathfrak{t}(f)\cong L^m\end{aligned}$$

as $\Bbbk$-vector spaces. The corresponding isomorphism $\varphi_m:L^m\to\mathrm{Hom}_{A_n^e}(P_m,A_n)$ is given by $(a,f)\mapsto\xi_{(a,f)}$, where $\xi_{(a,f)}(\mathfrak{o}(g)\otimes\mathfrak{t}(g))$ is a if $f=g$ and is 0 otherwise. Then the diagram

$$\begin{tikzcd}\cdots \arrow[r] & \Bbbk(\mathcal{B}//F^m) \arrow[r, "\sigma^{m+1}"] \arrow[d, "\varphi^m"] & \Bbbk\left(\mathcal{B}//F^{m+1}\right) \arrow[r] \arrow[d, "\varphi^{m+1}"] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}_{A_n^e}(P_m,A_n) \arrow[r, "d_{n+1}^*"] & \mathrm{Hom}_{A_n^e}(P_{m+1},A_n) \arrow[r] & \cdots\end{tikzcd}$$

is commutative. Therefore, the isomorphism of complexes is obtained. □

We now calculate the Hochschild cohomology groups $HH^m(A_n)$ by $\mathrm{Ker}\sigma^{m+1}/\mathrm{Im}\sigma^m$. Setting $e_1\prec e_2\prec\cdots\prec e_n\prec\alpha_1\prec\alpha_2\prec\cdots\prec\alpha_n$, and

$$(b,f_i^m)\prec(b',f_{i'}^m)\quad\text{if}\quad b\prec b',$$

for any $(b,f_i^m),(b',f_{i'}^m)\in\mathcal{B}//F^m$. We still denote by σ^m the matrix of σ^m under the ordered basis $\mathcal{B}//F^m$. Then

$$\sigma^{kn+1}=\begin{pmatrix}(-1)^{kn+1} & & & 1\\ 1 & (-1)^{kn+1} & & \\ & \ddots & \ddots & \\ & & 1 & (-1)^{kn+1}\end{pmatrix}_{n\times n},$$

and $\sigma^m=0$ if $m\neq kn+1$. Then, by direct calculation, we get a $\Bbbk$-basis of $HH^m(A_n)$ as following.

Proposition 3.1. *Let A_n be the self-injective quadratic monomial algebra. Then*

$$HH^m(A_n)\cong\begin{cases}\Bbbk\left\{\sum\limits_{i=1}^n(e_i,f_i^m)\right\}, & \text{if } m=kn \text{ is even;}\\ \Bbbk\left\{\sum\limits_{i=1}^n(\alpha_i,f_i^m)\right\}, & \text{if } m=kn+1 \text{ is odd;}\\ 0, & \text{otherwise.}\end{cases}$$

§4. Hochschild cohomology ring

In this section, the cup product of the cohomology ring $HH^*(A_n)$ is described by the parallel paths, and so that the ring structure of $HH^*(A_n)$ and $HH^*(A_n)/\mathcal{N}$ are given explicitly.

For any finite-dimensional $\Bbbk$-algebra Λ, Siegel and Witherspoon proved that any projective Λ^e-resolution $\mathbb{X}$ of Λ gives rise to the cup product on $HH^*(\Lambda)=\bigoplus_{m\geq0}HH^m(\Lambda)$ (see [11]). They showed that there exists a chain map $\triangle:\mathbb{X}\to\mathbb{X}\otimes_\Lambda\mathbb{X}$ lifting the identity, which is unique up to homotopy, and the cup product of two elements η in $HH^m(\Lambda)$ and ξ in $HH^n(\Lambda)$ can be defined by the composition of the maps

$$\mathbb{X} \xrightarrow{\triangle} \mathbb{X} \otimes_\Lambda \mathbb{X} \xrightarrow{\eta\otimes\xi} \Lambda \otimes_\Lambda \Lambda \xrightarrow{\omega} \Lambda.$$

where ω is the natural isomorphism.

Here, we will use the minimal projective bimodule resolution $\mathbb{P}=(P_m,d_m)$ of A_n which is constructed in Section 3, to give the cup product of $HH^*(A_n)$. First recall that the tensor complex $\mathbb{P}\otimes_{A_n}\mathbb{P}:=(\mathscr{P}_m,b_m)$ is given by

$$\mathscr{P}_m := \bigoplus_{i+j=m} P_i \otimes_{A_n} P_j,$$

and the differential $b_m : \mathscr{P}_m \to \mathscr{P}_{m-1}$ is given by

$$b_m = \sum_{i=0}^{m-1} \left((-1)^i \mathrm{id} \otimes d_{m-i} + d_{i+1} \otimes \mathrm{id}\right),$$

for all $m \geq 1$. It is well known that $\mathbb{P}\otimes_{A_n}\mathbb{P}$ is also a projective bimodule resolution of $A_n \cong A_n \otimes_{A_n} A_n$ Now we define a family of A_n^e-morphisms $\{\triangle_m : P_m \to \mathscr{P}_m\}_{m\geq 0}$ as follows:

$$\triangle_m(\mathfrak{o}(f_i^m)\otimes\mathfrak{t}(f_i^m)) = \sum_{s=0}^{m} (\mathfrak{o}(f_i^s)\otimes\mathfrak{t}(f_i^s)) \bar{\otimes} \left(\mathfrak{o}(f_{i+s}^{m-s})\otimes\mathfrak{t}(f_{i+s}^{m-s})\right),$$

where $\bar{\otimes} := \otimes_{A_n}$.

Lemma 4.1. *The morphism $\triangle := (\triangle_m)_{m\geq 0}$ satisfies the following commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & P_m \arrow[r, "d_m"] \arrow[d, "\triangle_m"] & P_{m-1} \arrow[r] \arrow[d, "\triangle_{m-1}"] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\triangle_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\triangle_0"] & A_n \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \mathscr{P}_m \arrow[r, "b_m"] & \mathscr{P}_{m-1} \arrow[r] & \cdots \arrow[r] & \mathscr{P}_1 \arrow[r, "b_1"] & \mathscr{P}_0 \arrow[r, "b_0"] & A_n \arrow[r] & 0
\end{tikzcd}$$

where $b_0 = \omega\circ(d_0\otimes d_0)$, d_0 is the multiplication map, $\omega : A_n\otimes_{A_n} A_n \to A_n$ is the natural isomorphism.

Proof. Firstly, it is easy to see that $d_0 = b_0 \circ \triangle_0$. Secondly, for $n=1$ and each α_i, we have

$$\begin{aligned}
b_1 \circ \triangle_1(\mathfrak{o}(\alpha_i)\otimes\mathfrak{t}(\alpha_i)) &= b_1\left((e_i\otimes e_i)\bar{\otimes}(e_i\otimes e_{i+1}) + (e_i\otimes e_{i+1})\bar{\otimes}(e_{i+1}\otimes e_{i+1})\right) \\
&= (e_i\otimes e_i)\bar{\otimes}(\alpha_i\otimes e_{i+1}) - (e_i\otimes e_i)\bar{\otimes}(e_i\otimes\alpha_i) \\
&\quad + (\alpha_i\otimes e_{i+1})\bar{\otimes}(e_{i+1}\otimes e_{i+1}) - (e_i\otimes\alpha_i)\bar{\otimes}(e_{i+1}\otimes e_{i+1}) \\
&= \triangle_0(\alpha_i\otimes\mathfrak{t}(\alpha_i) - \mathfrak{o}(\alpha_i)\otimes\alpha_i) \\
&= \triangle_0\circ d_1(\mathfrak{o}(\alpha_i)\otimes\mathfrak{t}(\alpha_i)).
\end{aligned}$$

Thus $\triangle_0\circ d_1 = b_1\circ\triangle_1$.

Finally, let $m\geq 2$. For any $a\in F^m$, $0\leq s\leq m$, we denote by $\triangle_{m-1}\circ d_m(\mathfrak{o}(a)\otimes\mathfrak{t}(a))_s$, $b_m\circ\triangle_m(\mathfrak{o}(a)\otimes\mathfrak{t}(a))_s$ the s-th direct summer of $\triangle_0\circ d_1(\mathfrak{o}(a)\otimes\mathfrak{t}(a))$ and $b_1\circ\triangle_1(\mathfrak{o}(a)\otimes\mathfrak{t}(a))$

respectively, i.e., $\triangle_0\circ d_1(\mathfrak{o}(a)\otimes\mathfrak{t}(a))_s\subseteq P_s\bar{\otimes}P_{m-s-1}$, $b_1\circ\triangle_1(\mathfrak{o}(a)\otimes\mathfrak{t}(a))_s\subseteq P_s\bar{\otimes}P_{m-s-1}$. Then, we have

$$\begin{aligned}b_m\circ\triangle_m(\mathfrak{o}(f_i^m)\otimes\mathfrak{t}(f_i^m))_s&=\big(\alpha_i\otimes\mathfrak{t}(f_{i+1}^s)\big)\bar{\otimes}\big(\mathfrak{o}(f_{i+s+1}^{m-s-1})\otimes\mathfrak{t}(f_{i+s+1}^{m-s-1})\big)\\&\quad+(-1)^m\big(\mathfrak{o}(f_i^s)\otimes\mathfrak{t}(f_i^s)\big)\bar{\otimes}\big(\mathfrak{o}(f_{i+s}^{m-s-1})\otimes\alpha_{i+m-1}\big)\\&=\triangle_{n-1}\circ d_m(\mathfrak{o}(f_i^m)\otimes\mathfrak{t}(f_i^m))_s,\end{aligned}$$

Therefore, we obtain the commutative diagram. □

Now, for any $m\geq 0$ and $\eta:=(a,f)\in L^m=\Bbbk\{\mathcal{B}//F^m\}$, we identify it with its image $\varphi_m(\eta)$ under the isomorphism $\varphi_m:L^m\to\mathrm{Hom}_{A_n^e}(P_m,A_n)$ which is given in Section 3. By the morphism $\triangle:=(\triangle_m)_{m\geq 0}$, the following theorem will give a description of the cup product using the parallel paths.

Proposition 4.1. *Suppose $\eta:=(a,f)\in HH^m(A_n)$ and $\xi:=(a',f')\in HH^l(A_n)$. Then*

$$\eta\sqcup\xi=\begin{cases}\big(aa',f_i^{m+l}\big), & \text{if } f=f_i^m \text{ and } f'=f_{i+m}^l;\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Let $f=f_i^m$ and $f'=f_{i'}^l$. Since the cup product of η and ξ is given by the composition of the maps $\mathbb{X}\xrightarrow{\triangle}\mathbb{X}\otimes_\Lambda\mathbb{X}\xrightarrow{\eta\otimes\xi}\Lambda\otimes_\Lambda\Lambda\xrightarrow{\omega}\Lambda$, we have $\eta\sqcup\xi=0$ if $i'\neq i+m$, and if $i'=i+m$, we have

$$(\eta\sqcup\xi)\big(\mathfrak{o}(f_i^{m+l})\otimes\mathfrak{t}(f_i^{m+l})\big)=\omega\Big(\sum_{s=0}^{m+l}\eta\big(\mathfrak{o}(f_i^s)\otimes\mathfrak{t}(f_i^s)\big)\bar{\otimes}\xi\big(\mathfrak{o}(f_{i+s}^{m+l-s})\otimes\mathfrak{t}(f_{i+s}^{m+l-s})\big)\Big)=aa'.$$

The proof is finished. □

Now, using the basis of $HH^m(A_n)$ in the pervious section and the description of the cup product in Proposition 4.1, we can give the ring structure of $HH^*(A_n)$.

Theorem 4.1. *As graded $\Bbbk$-algebras, we have the following isomorphism:*

$$\theta:\quad HH^*(A_n)\longrightarrow\Bbbk\langle y,u\rangle/I,$$

which is given by $\widehat{1}:=\sum_{i=1}^n(e_i,f_i^0)\mapsto 1$, $\widehat{y}:=\sum_{i=1}^n(\alpha_i,f_i^1)\mapsto y$, $\widehat{u}:=\sum_{i=1}^n(e_i,f_i^{kn})\mapsto u$, *where kn is even, I is the ideal of $\Bbbk\langle y,u\rangle$ generated by y^2 and $yu-uy$, and the degree of y and u is 1 and kn respectively.*

Proof. By using the formula given in Proposition 4.1, we can directly calculate that $\widehat{1}$ is the unit under the cup product, $\widehat{y}\sqcup\widehat{y}=0$, $\widehat{y}\sqcup\widehat{u}=\sum_{i=1}^n(\alpha_i,f_i^{kn+1})=\widehat{u}\sqcup\widehat{y}$. Thus, the correspondence in the theorem gives an isomorphism between graded algebras. □

For any finite-dimensional $\Bbbk$-algebra Λ, let $\mathcal{N}$ be the ideal of $HH^*(\Lambda)$ generated by all the homogeneous nilpotent elements. If $HH^*(\Lambda)/\mathcal{N}$ is a finite-dimensional commutative $\Bbbk$-algebra, then it is used to define the support varieties for Λ-modules [10]. Moreover, Snashall and Solberg conjectured that $HH^*(\Lambda)/\mathcal{N}$ is finitely generated for any finite-dimensional $\Bbbk$-algebra Λ. Here, using the result in Theorem 4.1, we can give the ring structure of $HH^*(A_n)/\mathcal{N}$ directly.

Corollary 4.1. *For the quotient algebra $HH^*(A_n)/\mathcal{N}$, we have*

$$HH^*(A_n)/\mathcal{N} \cong \Bbbk\langle u\rangle,$$

where the degree of u is kn and kn is even.

§5. Batalin-Vilkovisky structure on Hochschild cohomology

In this section, we give the Gerstenhaber algebra structure and the Batalin-Vilkovisky algebra structure on $HH^*(A_n)$ clearly.

Firstly, we consider the chain mappings between the reduced bar resolution and the minimal projective bimodule resolution of A_n. Recall that the algebra A_n has a reduced bar resolution $\bar{\mathbb{B}}=(\bar{B}_m,\bar{d}_m)$, where $\bar{B}_m=A_n\otimes_E \bar{A}_n^{\otimes_E m}\otimes_E A_n$, $\bar{A}_n=A_n/E$, E is the subalgebra of A_n generated by $\{e_1,e_2,\cdots,e_n\}$. On the other hand, we get a minimal projective bimodule resolution $\mathbb{P}=(P_m,d_m)$ of A_n in section 3. By using the method given in [1], we can give two chain mappings between $\mathbb{P}$ and $\bar{\mathbb{B}}$. We define $\Phi=(\Phi_m)_{m\geq 0}$ from $\mathbb{P}=(P_m,d_m)$ to $\bar{\mathbb{B}}=(\bar{B}_m,\bar{d}_m)$ by $\Phi_m:P_m\to\bar{B}_m$,

$$\Phi_m(\mathfrak{o}(f_i^m)\otimes\mathfrak{t}(f_i^m))=\mathfrak{o}(\alpha_i)\widehat{\otimes}\alpha_i\widehat{\otimes}\cdots\widehat{\otimes}\alpha_{i+m-1}\widehat{\otimes}\mathfrak{t}(\alpha_{i+m-1}),$$

for any $1\leq i\leq n$, where $\widehat{\otimes}:=\otimes_E$; define $\Psi=(\Psi_m)_{m\geq 0}$ from $\bar{\mathbb{B}}=(\bar{B}_m,\bar{d}_m)$ to $\mathbb{P}=(P_m,d_m)$ by $\Psi_m:\bar{B}_m\to P_m$,

$$\Psi_m(e_i\widehat{\otimes}\alpha_i\widehat{\otimes}\cdots\widehat{\otimes}\alpha_{i+m-1}\widehat{\otimes}e_{i+m})=\mathfrak{o}(f_i^m)\otimes\mathfrak{t}(f_i^m),$$

for any $1\leq i\leq n$. Then one can check that $\Phi=(\Phi_m)_{m\geq 0}$ and $\Psi=(\Psi_m)_{m\geq 0}$ are chain mappings and $\Psi_m\circ\Phi_m=\mathrm{id}_{P_m}$.

Since A_n is a self-injective algebra. If we define an bilinear form on A_n by

$$\langle a,b\rangle=\begin{cases}1, & \text{if } ab=\alpha_i, 1\leq i\leq n;\\ 0, & \text{overwise,}\end{cases}$$

then the corresponding semisimple Nakayama automorphism ν is given by

$a\in\mathcal{B}$	e_i	α_i
$\nu(a)$	e_{i+1}	α_{i+1}

where $e_{n+1}=e_1$ and $\alpha_{n+1}=\alpha_1$. In [8] and [13], the authors proved that the Hochschild cohomology ring of a Frobenius algebra with semisimple Nakayama automorphism is a Batalin-Vilkovisky algebra in different ways. For the algebra A_n, we can define an automorphism $\bar{(\,)}$ by

$a\in\mathcal{B}$	e_i	α_i
$\bar{a}$	α_i	e_{i+1}

then for any $a,b\in\mathcal{B}$, $\langle a,b\rangle=1$ if $b=\bar{a}$ and is 0 otherwise. Thus, we can calculate $\Delta(\alpha)$ by

$$\Delta(\alpha)(a_1\widehat{\otimes}\ldots\widehat{\otimes}a_{m-1})=\sum_{b\in\mathcal{B}_1}\Big\langle\sum_{i=1}^{m}\mu\alpha(a_i\widehat{\otimes}\ldots\widehat{\otimes}a_{m-1}\widehat{\otimes}\bar{b}\widehat{\otimes}\nu(a_1)\widehat{\otimes}\ldots\widehat{\otimes}\nu(a_{i-1})),1\Big\rangle b,$$

for any $\alpha\in HH^m(A_n)$, where $\mathcal{B}_1=\{e_i\,|\,1\le i\le n\}$ and $\mu=(-1)^{i(m-1)}$.

By the formulas

$$[f\sqcup g,h]=[f,h]\sqcup g+(-1)^{|f|(|h|-1)}f\sqcup[h,g],$$
$$[f,g]=-(-1)^{(|f|-1)|g|}(\Delta(f\sqcup g)-\Delta(f)\sqcup g-(-1)^{|f|}f\sqcup\Delta(g)),$$

we have

$$\Delta(f\sqcup g\sqcup h)=\Delta(f\sqcup g)\sqcup h+(-1)^{|g||h|}\Delta(f\sqcup h)\sqcup g+(-1)^{|f|+|g|+|h|}f\sqcup\Delta(g\sqcup h)$$
$$-\Delta(f)\sqcup g\sqcup h-(-1)^{|f||g|}f\sqcup h\sqcup\Delta(g)-(-1)^{|f|+|g|+|h|}f\sqcup\Delta(h)\sqcup g.$$

This means that to determine operator Δ, we only need to calculate $\Delta(a)$ and $\Delta(a\sqcup b)$ for all the generators a,b of $HH^*(A_n)$. Moreover, using the comparison morphisms Ψ and Φ, we compute $\Delta(f)$ by formula $\Delta(f)=\Delta(f\circ\Psi_m)\Phi_{m-1}$, for any $f\in HH^m(A_n)$.

Theorem 5.1. *Let A_n be the self-injective quadratic monomial algebra. Denote by Δ the Batalin-Vilkovisky operator on $HH^*\cong\Bbbk\langle y,u\rangle/I$. Then we have*

$$\Delta(y)=1,\qquad \Delta(u)=0,\qquad \Delta(yu)=(kn+1)u.$$

Proof. Firstly, note that

$$\begin{aligned}\Delta(f)(e_i\otimes e_i)&=\Delta(f\circ\Psi_1)\circ\Phi_0(e_i\otimes e_i)\\&=\big\langle f\circ\Psi_1(\mathfrak{o}(\alpha_i)\widehat{\otimes}\alpha_i\widehat{\otimes}\mathfrak{t}(\alpha_i)),1\big\rangle e_i\\&=\big\langle f(\mathfrak{o}(\alpha_i)\otimes\mathfrak{t}(\alpha_i)),1\big\rangle e_i,\end{aligned}$$

we get $\Delta(y)(e_i\otimes e_i)=0$ for $i=1,2,\cdots,n$. That is $\Delta(y)=1$. Secondly, since

$$\begin{aligned}\Delta(f)(\mathfrak{o}(f_i^{kn-1})\otimes\mathfrak{t}(f_i^{kn-1}))=&-\big\langle f\circ\Psi_{kn}(\mathfrak{o}(\alpha_i)\widehat{\otimes}\alpha_i\widehat{\otimes}\cdots\widehat{\otimes}\alpha_{i+kn-1}\widehat{\otimes}\mathfrak{t}(\alpha_{i+kn-1})),1\big\rangle e_{i+kn-1}\\&+\big\langle f\circ\Psi_{kn}(\mathfrak{o}(\alpha_i)\widehat{\otimes}\alpha_i\widehat{\otimes}\cdots\widehat{\otimes}\alpha_{i+kn-1}\widehat{\otimes}\mathfrak{t}(\alpha_{i+kn-1})),1\big\rangle e_i,\end{aligned}$$

we get $\Delta(u)(\mathfrak{o}(f_i^{kn-1})\otimes\mathfrak{t}(f_i^{kn-1}))=0$ for $i=1,2,\cdots,n$. That is $\Delta(u)=0$. Finally, by direct calculation,

$$\begin{aligned}\Delta(f)(\mathfrak{o}(f_i^{kn})\otimes\mathfrak{t}(f_i^{kn}))=&\sum_{i'=1}^{kn+1}\big\langle f\circ\Psi_{kn+1}(\mathfrak{o}(\alpha_{i+i'-1})\widehat{\otimes}\alpha_{i+i'-1}\widehat{\otimes}\cdots\widehat{\otimes}\alpha_{i+kn-1}\widehat{\otimes}\\&\alpha_i\widehat{\otimes}\nu(\alpha_i)\widehat{\otimes}\cdots\widehat{\otimes}\nu(\alpha_{i+i'-2})\widehat{\otimes}\mathfrak{t}(\nu(\alpha_{i+i'-2})),1\big\rangle e_i\\=&\sum_{i'=1}^{kn+1}\big\langle f(\mathfrak{o}(f_{i+i'-1}^{kn+1})\otimes\mathfrak{t}(f_{i+i'-1}^{kn+1})),1\big\rangle e_i.\end{aligned}$$

Thus $\Delta(yu)(\mathfrak{o}(f_i^{kn})\otimes\mathfrak{t}(f_i^{kn}))=(kn+1)e_i$ for $i=1,2,\cdots,n$. Hence $\Delta(yu)=(kn+1)u$. □

Using the Batalin-Vilkovisky operator Δ on $HH^*(A_n)$, we can determine the Gerstenhaber bracket [,] on $HH^*(A_n)$ by setting

$$[\alpha,\beta]=(-1)^{|\alpha||\beta|+|\alpha|+|\beta|}\Big((-1)^{|\alpha|+1}\Delta(\alpha\sqcup\beta)+(-1)^{|\alpha|}\Delta(\alpha)\sqcup\beta+\alpha\sqcup\Delta(\beta)\Big),$$

for any homogeneous elements $\alpha,\beta\in HH^*(A_n)$. Then the Gerstenhaber algebra structure on $HH^*(A_n)$ can be induced.

Corollary 5.1. *Let A_n be the self-injective quadratic monomial algebra. Then Gerstenhaber bracket on the $HH^*(A_n)\cong \Bbbk\langle y,u\rangle/I$ is induced by*

$$[y,y]=0,\qquad [u,u]=0,\qquad [y,u]=-knu.$$

Now we can give a complete description of the Batalin-Vilkovisky algebra structure on Hochschild cohomology of the self-injective quadratic monomial algebras.

Corollary 5.2. *Let A_n be the self-injective quadratic monomial algebra. The Batalin-Vilkovisky algebra $(HH^*(A_n),\ \sqcup,[\ ,\ ],\Delta)$ is isomorphic to $\Bbbk\langle y,u\rangle/I$, where the ideal I is generated by y^2 and $yu-uy$, the Gerstenhaber bracket is induced by $[y,y]=0$, $[u,u]=0$, $[y,u]=-knu$, the Batalin-Vilkovisky operator is induced by $\Delta(y)=1$, $\Delta(u)=0$, $\Delta(yu)=(kn+1)u$, and the degree of y and u is 1 and kn respectively, kn is even.*

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