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# $\mathcal{Y}$-Gorenstein Cotorsion Modules

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**Abstract**: Let $R$ be an associative ring with identity, and $\mathcal{Y}$ be a class of right $R$-modules, which contains all injective right $R$-modules. In this paper, we introduce the definition of $\mathcal{Y}$-Gorenstein cotorsion modules, which is a generalization of cotorsion and Gorenstein cotorsion modules. We discuss the relationship between Gorenstein cotorsion, weakly Gorenstein cotorsion and $\mathcal{Y}$-Gorenstein cotorsion modules. We investigate properties and characterizations of $\mathcal{Y}$-Gorenstein cotorsion modules.

**Keywords**: $\mathcal{Y}$-Gorenstein cotorsion; $\mathcal{Y}$-Gorenstein flat; Strongly Gorenstein flat; Weakly Gorenstein cotorsion; Coherent ring

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## §1. Introduction

Gorenstein homological theory plays an important role in relative homological algebra. In 2001, Enochs et al [5] introduced Gorenstein flat modules over an associative ring $R$, which is an important generalization of flat modules. Lei and Meng [7] investigated Gorenstein cotorsion modules relative to Gorenstein flat modules, and got many properties similar to cotorsion modules. Ding et al [1] introduced and invstigated strongly Gorenstein flat modules. Based on the above, Zhao et al [12] discussed weakly Gorenstein cotorsion modules relative to strongly Gorenstein flat modules. Meng [9] gave the notion of $\mathcal{Y}$-Gorenstein flat modules, in which $\mathcal{Y}$ is a class of right $R$-modules and contains all injective right $R$-modules. As is well known, flat modules are $\mathcal{Y}$-Gorenstein flat, and $\mathcal{Y}$-Gorenstein flat modules are Gorenstein flat. When $\mathcal{Y}$ is the class of all injective right $R$-modules, $\mathcal{Y}$-Gorenstein flat modules are precisely Gorenstein flat modules. Naturally, we can consider cotorsion modules relative to $\mathcal{Y}$-Gorenstein flat modules, which will be called $\mathcal{Y}$-Gorenstein cotorsion modules in the paper. Furthermore, we investigate the relationship between Gorenstein cotorsion modules, weakly Gorenstein cotorsion modules

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and $\mathcal{Y}$-Gorenstein cotorsion modules.

We now state main results of this paper.

**Theorem 1.1.** *Let $\mathcal{F}$ be the class of all flat left $R$-modules, and $\mathcal{Y}$ be a class of right $R$-modules which contains all injective right $R$-modules. If $\mathcal{Y}^+ \subseteq \mathcal{F}$, then every $\mathcal{Y}$-Gorenstein cotorsion left $R$-module is weakly Gorenstein cotorsion.*

**Theorem 1.2.** *The following statements hold:*

(1) *For a family $\{M_i\}_{i\in I}$ of $R$-modules, all $M_i$ are $\mathcal{Y}$-Gorenstein cotorsion if and only if $\prod_{i\in I} M_i$ is $\mathcal{Y}$-Gorenstein cotorsion.*

(2) *Let $0\to M\to N\to L\to 0$ be an exact sequence of left $R$-modules. If $M,L$ are $\mathcal{Y}$-Gorenstein cotorsion modules, then $N$ is also $\mathcal{Y}$-Gorenstein cotorsion.*

**Theorem 1.3.** *Let $R$ be a right coherent ring, every left $R$-module has finite $\mathcal{Y}$-Gorenstein flat dimension, and $\mathcal{A}$ be a nonempty collection of left ideals of $R$. Then the following statements are equivalent:*

(1) *Every $\mathcal{Y}$-Gorenstein cotorsion module ${}_RM$ is $\mathcal{A}$-injective.*

(2) *$R/A$ is $\mathcal{Y}$-Gorenstein flat for any $A\in\mathcal{A}$.*

**Theorem 1.4.** *Let $R$ be a left perfect ring. Then the following statements are equivalent:*

(1) *Every left $R$-module is $\mathcal{Y}$-Gorenstein cotorsion.*

(2) *Every $\mathcal{Y}$-Gorenstein flat left $R$-module is $\mathcal{Y}$-Gorenstein cotorsion.*

This paper is organized as follows. Section 2 contains some known notions and results for use throughout the paper, and we introduce $\mathcal{Y}$-Gorenstein cotorsion modules, and discuss the relationship between Gorenstein cotorsion modules, weakly Gorenstein cotorsion modules and $\mathcal{Y}$-Gorenstein cotorsion modules. Section 3 is devoted to properties and characterizations of $\mathcal{Y}$-Gorenstein cotorsion modules.

## §2. Preliminaries

Throughout the paper, $R$ will be an associative ring with identity and all modules are unital $R$-modules. We use $R$-Mod (resp., Mod-$R$) to denote the class of left (resp., right) $R$-modules. ${}_RM$ (resp., $M_R$) denotes a left (resp., right) $R$-module $M$. A left $R$-module $C$ is called cotorsion if $\mathrm{Ext}^1_R(F,C)=0$ for any flat left $R$-module $F$ (see [8]). Following [5], a module ${}_RM$ is said to be Gorenstein flat, if there exists an exact sequence of flat left $R$-modules $\cdots\to F_1\to F_0\to F^0\to F^1\to\ldots$ with $M\cong\mathrm{Ker}(F^0\to F^1)$, such that $E\otimes_R -$ leaves the sequence exact for every injective right $R$-module $E$. A module ${}_RN$ is called Gorenstein cotorsion if $\mathrm{Ext}^1_R(F,N)=0$ for any Gorenstein flat left $R$-module $F$ (see [7]). In addition, a module ${}_RM$ is said to be strongly Gorenstein flat, if there exists an exact sequence of projective left $R$-modules $\cdots\to P_1\to P_0\to P^0\to P^1\to\ldots$ with $M\cong\mathrm{Ker}(P^0\to P^1)$, such that $\mathrm{Hom}_R(-,F)$ leaves the sequence exact for every flat left $R$-module $F$. When $R$ is a right coherent ring, every strongly Gorenstein flat left $R$-module is Gorenstein flat. A module ${}_RN$ is called weakly Gorenstein cotorsion if $\mathrm{Ext}^1_R(F,N)=0$ for any strongly Gorenstein flat left $R$-module $F$ (see [12]).

Let $\mathcal{X}$ be a class of left $R$-modules that contains all projective left $R$-modules, and $\mathcal{Y}$ be a class of right $R$-modules that contains all injective right $R$-modules. According to [9], a module ${}_RM$ is said to be $\mathcal{X}$-Gorenstein projective, if there exists an exact sequence of projective left $R$-modules $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \dots$ with $M \cong \mathrm{Ker}(P^0 \to P^1)$, such that $\mathrm{Hom}_R(-,X)$ leaves the sequence exact for every $X \in \mathcal{X}$. A module ${}_RN$ is called $\mathcal{Y}$-Gorenstein flat if there exists an exact sequence of flat left $R$-modules $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \dots$ with $M \cong \mathrm{Ker}(F^0 \to F^1)$, such that $Y \otimes_R -$ leaves the sequence exact for every $Y \in \mathcal{Y}$.

**Remark 2.1.** (1) *$\mathcal{Y}$-Gorenstein flat modules are Gorenstein flat.*

(2) *If $\mathcal{Y}$ is the class of all injective right $R$-modules, then Gorenstein flat modules are precisely $\mathcal{Y}$-Gorenstein flat modules.*

(3) *If $\mathcal{X}$ is the class of all flat modules, then the class of $\mathcal{X}$-Gorenstein projective modules coincides with that of strongly Gorenstein flat modules.*

Let $\mathcal{A}$ be a nonempty collection of left ideals of a ring $R$. ${}_RM$ is said to be $\mathcal{A}$-injective, if every $R$-homomorphism $f: A \to M$ with $A \in \mathcal{A}$ can be lifted to an $R$-homomorphism $g: R \to M$, or equivalently $\mathrm{Ext}^1_R(R/A, M) = 0$ for any $A \in \mathcal{A}$ (see [11]).

Let $\mathcal{C}$ be a class of left $R$-module (closed under isomorphisms). A $\mathcal{C}$-precover of ${}_RM$ is a morphism $\varphi: C \to M$ with $C \in \mathcal{C}$ such that $\mathrm{Hom}_R(C^{'}, \varphi)$ is surjective for every $C^{'} \in \mathcal{C}$. If in addition, any morphism $\alpha: C \to C$ verifying $\varphi \circ \alpha = \varphi$ is an automorphism, then $\varphi$ is said to be a $\mathcal{C}$-cover of ${}_RM$ (see [2]). Denote $\mathcal{C}^{\perp} = \{M \in R\text{-Mod} \mid \mathrm{Ext}^1_R(C, M) = 0 \text{ for any } C \in \mathcal{C}\}$, and ${}^{\perp}\mathcal{C} = \{N \in R\text{-Mod} \mid \mathrm{Ext}^1_R(N, C) = 0 \text{ for any } C \in \mathcal{C}\}$. A pair $(\mathcal{F}, \mathcal{C})$ of classes of left $R$-modules is called a cotorsion theory (which is also called a cotorsion pair), if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$ (see [3]). A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is complete, if for any $M \in R$-Mod, there exist short exact sequences $0 \to C \to F \to M \to 0$ and $0 \to M \to C^{'} \to F^{'} \to 0$ with $F, F^{'} \in \mathcal{F}$ and $C, C^{'} \in \mathcal{C}$ (see [10]). A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is hereditary, if whenever $0 \to L^{'} \to L \to L^{''} \to 0$ is exact with $L, L^{''} \in \mathcal{F}$, then $L^{'}$ is also in $\mathcal{F}$ (see [4,6]).

**Definition 2.1.** *Let $\mathcal{Y}$ be a class of right $R$-modules that contains all injective right $R$-modules. A module ${}_RM$ is said to be $\mathcal{Y}$-Gorenstein cotorsion provided that $\mathrm{Ext}^1_R(Q, M) = 0$ for any $\mathcal{Y}$-Gorenstein flat left $R$-module $Q$.*

It is obvious that cotorsion modules and Gorenstein cotorsion modules are $\mathcal{Y}$-Gorenstein cotorsion. If $\mathcal{Y}$ is the class of all injective right $R$-modules, then the class of Gorenstein cotorsion left $R$-modules coincides with that of $\mathcal{Y}$-Gorenstein cotorsion left $R$-modules.

**Lemma 2.1.** [9]*Let $\mathcal{X}$ be a class of left $R$-modules that contains all projective left $R$-modules, and $\mathcal{Y}$ be a class of right $R$-modules that contains all injective right $R$-modules. If $\mathcal{Y}^+ \subseteq \mathcal{X}$, where $\mathcal{Y}^+ = \{Y^+ = \mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Q}/\mathbb{Z}) \mid Y \in \mathcal{Y}\}$, then every $\mathcal{X}$-Gorenstein projective left $R$-module is $\mathcal{Y}$-Gorenstein flat.*

**Proposition 2.1.** *Let $\mathcal{F}$ be the class of all flat left $R$-modules, and $\mathcal{I}$ be the class of all injective right $R$-modules. If $\mathcal{I}^+ \subseteq \mathcal{F}$, then any $\mathcal{F}$-Gorenstein projective left $R$-module is $\mathcal{I}$-Gorenstein flat, which means strongly Gorenstein flat modules are Gorenstein flat.*

**Proposition 2.2.** *Let $\mathcal{F}$ be the class of all flat left $R$-modules, and $\mathcal{Y}$ be a class of right $R$-modules that contains all injective right $R$-modules. If $\mathcal{Y}^+ \subseteq \mathcal{F}$, then every strongly Gorenstein flat left $R$-module is $\mathcal{Y}$-Gorenstein flat.*

**Proof**. This is not difficult to prove by Remark 2.1(3) and Lemma 2.1. □

Following the above proposition, we can get the relationship between $\mathcal{Y}$-Gorenstein cotorsion left $R$-modules and weakly Gorenstein cotorsion modules.

**Theorem 2.1.** *Let $\mathcal{F}$ be the class of all flat left $R$-modules, and $\mathcal{Y}$ be a class of right $R$-modules which contains all injective right $R$-modules. If $\mathcal{Y}^+ \subseteq \mathcal{F}$, then every $\mathcal{Y}$-Gorenstein cotorsion left $R$-module is weakly Gorenstein cotorsion.*

According to definition of weakly Gorenstein cotorsion and $\mathcal{Y}$-Gorenstein cotorsion modules, it is easy to obtain that every $\mathcal{Y}$-Gorenstein cotorsion left $R$-module is weakly Gorenstein cotorsion.

**Corollary 2.1.** *If $R$ is a right coherent ring, then every Gorenstein cotorsion left $R$-module is weakly Gorenstein cotorsion.*

## §3. Properties and characterizations of $\mathcal{Y}$-Gorenstein cotorsion modules

In this section, $\mathcal{Y}$ denotes a class of right $R$-modules that contains all injective modules.

**Theorem 3.1.** *The following statements hold:*

(1) *For a family $\{M_i\}_{i\in I}$ of $R$-modules, all $M_i$ are $\mathcal{Y}$-Gorenstein cotorsion if and only if $\prod_{i\in I} M_i$ is $\mathcal{Y}$-Gorenstein cotorsion.*

(2) *Let $0 \to M \to N \to L \to 0$ be an exact sequence of left $R$-modules. If $M, L$ are $\mathcal{Y}$-Gorenstein cotorsion modules, then $N$ is also $\mathcal{Y}$-Gorenstein cotorsion.*

**Proof**. (1) Let $\{M_i\}_{i\in I}$ be a family of $R$-modules. One hand, if all $M_i$ are $\mathcal{Y}$-Gorenstein cotorsion left $R$-modules, then for any $\mathcal{Y}$-Gorenstein flat left $R$-module $Q$, we have the isomorphism

$$\mathrm{Ext}_R^1(Q, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Ext}_R^1(Q, M_i).$$

By assumption and Definition 2.1, we get $\mathrm{Ext}_R^1(Q, M_i) = 0$ for any $i \in I$. So $\mathrm{Ext}_R^1(Q, \prod_{i\in I} M_i) = 0$. Therefore $\prod_{i\in I} M_i$ is also $\mathcal{Y}$-Gorenstein cotorsion.

On the other hand, if $\prod_{i\in I} M_i$ is $\mathcal{Y}$-Gorenstein cotorsion, then for any $\mathcal{Y}$-Gorenstein flat left $R$-module $Q$, we have $\mathrm{Ext}_R^1(Q, \prod_{i\in I} M_i) = 0$. so $\mathrm{Ext}_R^1(Q, M_i)$=0 for any $i \in I$ by the above isomorphism. We can get that all $M_i$ are $\mathcal{Y}$-Gorenstein cotorsion modules.

(2) Assume that $M, L$ are $\mathcal{Y}$-Gorenstein cotorsion modules. For any $\mathcal{Y}$-Gorenstein flat left $R$-module $Q$, by applying the functor $\mathrm{Hom}_R(Q, -)$ to the sequence $0 \to M \to N \to L \to 0$, we can get the following long exact sequence

$$\cdots \to \mathrm{Hom}_R(Q, L) \to \mathrm{Ext}_R^1(Q, M) \to \mathrm{Ext}_R^1(Q, N) \to \mathrm{Ext}_R^1(Q, L) \to \ldots.$$

Since $M,L$ are $\mathcal{Y}$-Gorenstein cotorsion modules, we have $\mathrm{Ext}_R^1(Q,M)=0=\mathrm{Ext}_R^1(Q,L)$, so $\mathrm{Ext}_R^1(Q,N)=0$. It is easy to obtain that $N$ is also $\mathcal{Y}$-Gorenstein cotorsion. □

**Lemma 3.1.** *If $R$ is a right coherent ring, then the class of $\mathcal{Y}$-Gorenstein flat left $R$-modules are closed under extensions, kernels of epimorphisms, direct sums and direct summands.*

**Theorem 3.2.** *Let $R$ be a right coherent ring and ${}_RM$ be a $\mathcal{Y}$-Gorenstein cotorsion module. Then*

$$\mathrm{Ext}_R^i(Q,M)=0,$$

*for any $\mathcal{Y}$-Gorenstein flat left $R$-module $Q$ and any $i\geq 1$.*

**Proof**. For any $\mathcal{Y}$-Gorenstein flat left $R$-module $Q$, suppose $\cdots\to P_2\to P_1\to P_0\to Q\to 0$ is a projective resolution of $Q$. Let $K_0=\mathrm{Ker}(P_0\to Q)$, $K_j=\mathrm{Ker}(P_j\to P_{j-1})$ $(j\geq 1)$. Note that projective modules are $\mathcal{Y}$-Gorenstein flat, so $P_i$ is $\mathcal{Y}$-Gorenstein flat for any $i\geq 0$. Following Lemma 3.1, we can get $K_j$ is $\mathcal{Y}$-Gorenstein flat for any $j\geq 0$. By applying the functor $\mathrm{Hom}_R(-,M)$ to above resolution, it is not difficult to obtain that

$$\mathrm{Ext}_R^{j+1}(Q,M)\cong \mathrm{Ext}_R^1(K_{j-1},M)=0,$$

which means $\mathrm{Ext}_R^i(Q,M)=0$ for any $\mathcal{Y}$-Gorenstein flat left $R$-module $Q$ and any $i\geq 2$. The situation $i=1$ follows from Definition 2.1.□

**Proposition 3.1.** *Let $R$ be a right coherent ring and $(\varepsilon):0\to M\to N\to L\to 0$ be an exact sequence of left $R$-modules. If $N$ is $\mathcal{Y}$-Gorenstein cotorsion, then the following conditions are equivalent:*

(1) *$M$ is a $\mathcal{Y}$-Gorenstein cotorsion module.*

(2) *$(\varepsilon)$ stays exact under* $\mathrm{Hom}_R(Q,-)$ *for any $\mathcal{Y}$-Gorenstein flat module $Q$.*

**Proof**. $(1)\Rightarrow(2)$ It is obvious.

$(2)\Rightarrow(1)$ For any $\mathcal{Y}$-Gorenstein flat module $Q$, By applying the functor $\mathrm{Hom}_R(Q,-)$ to the sequence $(\varepsilon)$, we can get the following long exact sequence

$$\cdots\to \mathrm{Hom}_R(Q,L)\to \mathrm{Ext}_R^1(Q,M)\to \mathrm{Ext}_R^1(Q,N)\to \ldots.$$

Since $N$ is $\mathcal{Y}$-Gorenstein cotorsion, we have $\mathrm{Ext}_R^1(Q,N)=0$. Note that $(\varepsilon)$ stays exact under $\mathrm{Hom}_R(Q,-)$, so we can get $\mathrm{Ext}_R^1(Q,M)=0$. By Definition 2.1, it is easy to know that $M$ is a $\mathcal{Y}$-Gorenstein cotorsion module.□

**Corollary 3.1.** *Let $M$ be a right $R$-module with finite injective dimension. Then* $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ *is $\mathcal{Y}$-Gorenstein cotorsion module.*

**Proof**. Since Gorenstein cotorsion modules are $\mathcal{Y}$-Gorenstein cotorsion, it is not difficult to prove above conclusion by Proposition 2.4 in [7].□

Denote by $\mathcal{YGF}$ (resp., $\mathcal{YGC}$) the class of all $\mathcal{Y}$-Gorenstein flat (resp., cotorsion) left $R$-modules. By ( [9],Theorem 4.13), we can get the following result.

**Lemma 3.2.** *If $R$ is a right coherent ring and every left $R$-module has finite $\mathcal{Y}$-Gorenstein flat dimension. Then $(\mathcal{YGF},\mathcal{YGC})$ is a complete hereditary cotorsion theory.*

**Theorem 3.3.** *Let $R$ be a right coherent ring, every left $R$-module has finite $\mathcal{Y}$-Gorenstein flat dimension, and $\mathcal{A}$ be a nonempty collection of left ideals of $R$. Then the following statements are equivalent:*

(1) *Every $\mathcal{Y}$-Gorenstein cotorsion module ${}_RM$ is $\mathcal{A}$-injective.*

(2) *$R/A$ is $\mathcal{Y}$-Gorenstein flat for any $A \in \mathcal{A}$.*

**Proof**. (1)$\Rightarrow$(2) For any $A \in \mathcal{A}$, it is sufficient to prove $\mathrm{Ext}_R^1(R/A, C) = 0$ for every $\mathcal{Y}$-Gorenstein cotorsion module $C$. Consider the exact sequence $0 \to A \to R \to R/A \to 0$ and induced exact sequence

$$\cdots \to \mathrm{Hom}_R(R, C) \to \mathrm{Hom}_R(A, C) \to \mathrm{Ext}_R^1(R/A, C) \to \mathrm{Ext}_R^1(R, C) \to \ldots,$$

by (1) and $\mathrm{Ext}_R^1(R, C) = 0$, we can get $\mathrm{Ext}_R^1(R/A, C) = 0$ for every $\mathcal{Y}$-Gorenstein cotorsion module $C$. According to Lemma 3.2, we obtain that $R/A$ is $\mathcal{Y}$-Gorenstein flat for any $A \in \mathcal{A}$.

(2)$\Rightarrow$(1) For any $\mathcal{Y}$-Gorenstein cotorsion module ${}_RM$, it is only to prove $\mathrm{Ext}_R^1(R/A, M) = 0$ for any $A \in \mathcal{A}$, which means ${}_RM$ is $\mathcal{A}$-injective. By (2) and Definition 2.1, it is clear.□

**Lemma 3.3.** (1) *Let $\varphi: X \to M$ be an $\mathcal{X}$-cover of $M$, and assume that $\mathcal{X}$ is closed under extensions. Set $K = \mathrm{Ker}\varphi$. Then $\mathrm{Ext}_R^1(X', K) = 0$ for any $X' \in \mathcal{X}$.*

(2) *Let $\phi: M \to X$ be an $\mathcal{X}$-envelope of $M$, and assume that $\mathcal{X}$ is closed under extensions. Set $D = \mathrm{Coker}\phi = X/\phi(M)$. Then $\mathrm{Ext}_R^1(D, X') = 0$ for any $X' \in \mathcal{X}$.*

**Theorem 3.4.** *Let $R$ be a right coherent ring, and $Q$ be a $\mathcal{Y}$-Gorenstein flat cover of ${}_RM$ with $K = \mathrm{Ker}(Q \to M)$. Then $K$ is a $\mathcal{Y}$-Gorenstein cotorsion module. Moreover, if $M$ is a $\mathcal{Y}$-Gorenstein cotorsion module, then $Q$ is also a $\mathcal{Y}$-Gorenstein cotorsion module.*

**Proof**. According to Lemma 3.1 and Lemma 3.3 (1), it can be obtained that $\mathrm{Ext}_R^1(X', K) = 0$ for any $\mathcal{Y}$-Gorenstein flat module $X'$. Then $K$ is a $\mathcal{Y}$-Gorenstein cotorsion module by Definition 2.1. Moreover, if $M$ is a $\mathcal{Y}$-Gorenstein cotorsion module, then $Q$ is also a $\mathcal{Y}$-Gorenstein cotorsion module by properties of the Ext functor.□

**Theorem 3.5.** *Let $R$ be a right coherent ring, and $H$ be a $\mathcal{Y}$-Gorenstein cotorsion envelope of ${}_RM$ with $D = \mathrm{Coker}(M \to H)$. Then $C$ is a $\mathcal{Y}$-Gorenstein flat module. Moreover, if $M$ is a $\mathcal{Y}$-Gorenstein flat module, then $H$ is also a $\mathcal{Y}$-Gorenstein flat module.*

**Proof**. It is not difficult to prove by Theorem 3.1 (2) and Lemma 3.3 (2).□

**Proposition 3.2.** *Let $R$ be a commutative ring and $P$ a flat $R$-module. If $M$ is a $\mathcal{Y}$-Gorenstein flat module, then $M \otimes_R P$ is a $\mathcal{Y}$-Gorenstein flat module.*

**Proof**. Assume that $M$ is a $\mathcal{Y}$-Gorenstein flat module, then there exists an exact sequence of flat $R$-modules

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$$

with $M \cong \mathrm{Ker}(F^0 \to F^1)$ such that $Y \otimes_R -$ leaves the sequence exact for any $Y \in \mathcal{Y}$. Since $P$ is a flat module, we have exact sequence

$$\cdots \to F_1 \otimes_R P \to F_0 \otimes_R P \to F^0 \otimes_R P \to F^1 \otimes_R P \to \ldots$$

with $M\otimes_R P\cong \mathrm{Ker}(F^0\otimes_R P\to F^1\otimes_R P)$. Note that $F_i, F^i$ are flat for all $i\geq 0$ and $P$ is a flat module, we know that $F_i\otimes_R P$, $F^i\otimes_R P$ are flat modules. For any $Y\in\mathcal{Y}$, consider the following commutative diagram (Diagram 1)

$$\begin{tikzcd}
\cdots \arrow[r] & Y\otimes_R(F_0\otimes_R P) \arrow[r] \arrow[d, "\cong"] & Y\otimes_R(F^0\otimes_R P) \arrow[r] \arrow[d, "\cong"] & Y\otimes_R(F^1\otimes_R P) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & (Y\otimes_R F_0)\otimes_R P \arrow[r] & (Y\otimes_R F^0)\otimes_R P \arrow[r] & (Y\otimes_R F^1)\otimes_R P \arrow[r] & \cdots
\end{tikzcd}$$

Diagram 1 the constructed commutative diagram

Since $M$ is $\mathcal{Y}$-Gorenstein flat and $P$ is flat, it is not difficult to obtain that the upper sequence of commutative diagram is exact. Therefore $M\otimes_R P$ is a $\mathcal{Y}$-Gorenstein flat module.□

**Theorem 3.6.** *Let $R$ be a left perfect ring. Then the following statements are equivalent:*

(1) *Every left $R$-module is $\mathcal{Y}$-Gorenstein cotorsion.*

(2) *Every $\mathcal{Y}$-Gorenstein flat left $R$-module is $\mathcal{Y}$-Gorenstein cotorsion.*

**Proof**. (1)$\Rightarrow$(2) It is obvious.

(2)$\Rightarrow$(1) For any left $R$-module $M$, it is only to prove $\mathrm{Ext}^1_R(Q,M)=0$ for any $\mathcal{Y}$-Gorenstein flat module $Q$. By the definition of the $\mathcal{Y}$-Gorenstein flat modules, there is an exact sequence of flat left $R$-modules

$$\cdots\to F_1\to F_0\to F^0\to F^1\to\cdots$$

with $Q\cong\mathrm{Ker}(F^0\to F^1)$ such that $Y\otimes_R -$ leaves the sequence exact for any $Y\in\mathcal{Y}$. Let $K=\mathrm{Ker}(F^1\to F^2)$, consider the short exact sequence $0\to Q\to F^0\to K\to 0$, it is clear that $K$ is $\mathcal{Y}$-Gorenstein flat. Following (2), we know $Q$ is $\mathcal{Y}$-Gorenstein cotorsion, and $\mathrm{Ext}^1_R(K,Q)=0$. By applying the functor $\mathrm{Hom}_R(K,-)$ to the sequence, we can get exact sequence

$$0\to\mathrm{Hom}_R(K,Q)\to\mathrm{Hom}_R(K,F^0)\to\mathrm{Hom}_R(K,K)\to 0.$$

So $Q$ is direct summand of $F^0$. Since the class of all flat modules is closed under direct summands, $Q$ is flat. Note that $R$ is a left perfect, the class of all flat modules coincides with that of all projective modules, we have that $Q$ is projective. So $\mathrm{Ext}^1_R(Q,M)=0$. Therefore, Every $\mathcal{Y}$-Gorenstein flat left $R$-module is $\mathcal{Y}$-Gorenstein cotorsion.□

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